Supplementary Notes for MM08 Geometry I

6. Hopf's Umlaufsatz

Andrew Swann

FOR THE proof of the first version of the Gauß-Bonnet theorem in Pressley [3, Theorem 11.1] one needs a result known as Hopf's Umlaufsatz, see pages 250–251. These notes prove this theorem for curves in the plane and show how it may be applied to give a version of Theorem 11.1 in Pressley. Some of this material is based on §5-7 of do Carmo [1], another useful reference in preparing these notes was the lecture notes of Newhouse [2].

6.1 Rotation numbers in the plane

CONSIDER a regular closed plane curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ of period ℓ. Recall this means that γ is smooth, with $\gamma'(t) \neq 0$ for all t and

$$\gamma(t) = \gamma(t + n\ell) \qquad \text{for all } n \in \mathbb{Z},\ t \in \mathbb{R}.$$

The following Lemma enables to make a smooth choice of function φ that measures the angle between $\dot\gamma$ and a fixed vector.

Lemma 6.1. *Suppose* $c, s\colon \mathbb{R} \to \mathbb{R}$ *are smooth functions such that* $c^2 + s^2 = 1$. *Given* θ_0 *such that* $(\cos\theta_0, \sin\theta_0) = (c(0), s(0))$, *the function* $\theta\colon \mathbb{R} \to \mathbb{R}$ *defined by*

$$\theta(t) = \int_0^t (c\dot s - s\dot c)\,dt + \theta_0$$

is smooth and satisfies $c(t) = \cos(\theta(t))$, $s(t) = \sin(\theta(t))$ *for all* $t \in \mathbb{R}$.

Proof. As θ is the integral of a smooth function, it is itself smooth. Consider

$$F = \tfrac{1}{2}\big((c - \cos\theta)^2 + (s - \sin\theta)^2\big)$$

which is zero precisely when $(c, s) = (\cos\theta, \sin\theta)$. We have $F(0) = 0$, so it is sufficient to show that F is constant. Note that $c^2+s^2 = 1$ implies $c\dot{c}+s\dot{s} = 0$. We now compute

$$\begin{aligned}\dot{F} &= (\dot{c} + \dot{\theta}\sin\theta)(c - \cos\theta) + 2(\dot{s} - \dot{\theta}\cos\theta)(s - \sin\theta)\\ &= (\dot{c}c + \dot{s}s) - (\dot{c}\cos\theta + \dot{s}\sin\theta) + \dot{\theta}(c\sin\theta - s\cos\theta) + 0\\ &= 0 - (c^2 + s^2)(\dot{c}\cos\theta + \dot{s}\sin\theta) + (c\dot{s} - s\dot{c})(c\sin\theta - s\cos\theta)\\ &= \cos\theta\,(-c^2\dot{c} - c\,s\dot{s}) + \sin\theta\,(-s^2\dot{s} - s\,c\dot{c}) = 0,\end{aligned}$$

as required. ☐

For our curve γ, make a choice φ_0 for the angle between $\dot{\gamma}(0)$ and $\mathbf{e}_1$, i.e.,

$$\dot{\gamma}(0) = \|\dot{\gamma}(0)\|\,(\cos\varphi_0\,\mathbf{e}_1 + \sin\varphi_0\,\mathbf{e}_2).$$

The difference between two choices of φ_0 will be an integer multiple of 2π. Using Lemma 6.1 with $\dot{\gamma}/\|\dot{\gamma}\| = c\mathbf{e}_1 + s\mathbf{e}_2$, we see that

$$\varphi(t) = \int_0^t \frac{\langle \dot{\gamma} \times \ddot{\gamma}, \mathbf{e}_3\rangle}{\|\dot{\gamma}\|^2}\,dt + \varphi_0$$

is a smooth determination of the angle between $\dot{\gamma}$ and $\mathbf{e}_1$ along the curve. In particular, $\varphi(\ell)$ again measures the angle between $\dot{\gamma}(\ell) = \dot{\gamma}(0)$ and $\mathbf{e}_1$, so differs from φ_0 by an integer multiple of 2π.

Definition 6.2. The *rotation index* of γ is

$$\operatorname{ind}(\gamma) = \frac{1}{2\pi}(\varphi(\ell) - \varphi(0)) \in \mathbb{Z}.$$

Note that

$$\operatorname{ind}(\gamma) = \frac{1}{2\pi}\int_0^\ell \frac{\langle \dot{\gamma} \times \ddot{\gamma}, \mathbf{e}_3\rangle}{\|\dot{\gamma}\|^2}\,dt \tag{6.1}$$

so is independent of the choice of φ_0.

Definition 6.3. Suppose γ_0 and γ_1 are two regular closed plane curves of period ℓ. A smooth map $F\colon \mathbb{R} \times [0, 1] \to \mathbb{R}^2$ is an *isotopy* from γ_0 to γ_1 if

(i) $F(t, 0) = \gamma_0(t)$,

(ii) $F(t, 1) = \gamma_1(t)$,

(iii) $t \mapsto F(t, s)$ is a regular closed plane curve of period ℓ for each $s \in [0, 1]$.

We then say that γ_0 is *isotopic* to γ_1.

Lemma 6.4. *If γ_0 is isotopic to γ_1 then*

$$\operatorname{ind}(\gamma_0) = \operatorname{ind}(\gamma_1).$$

Proof. Let F be an isotopy from γ_0 to γ_1 and put $\gamma_s(t) = F(t, s)$. Then $s \mapsto \operatorname{ind}(\gamma_s)$ as given by equation (6.1) is a continuous function $[0, 1] \to \mathbb{Z}$, so constant. $\square$

One example of this is the isotopy $F(t, s) = \gamma(t + t_0 s)$, for $t_0 \in \mathbb{R}$ constant, which shows us that $\operatorname{ind}(\gamma)$ is independent of reparameterisations $t \mapsto t + t_0$, so

$$\operatorname{ind}(\gamma) = \frac{1}{2\pi}\int_{t_0}^{t_0+\ell} \frac{\langle \dot\gamma \times \ddot\gamma, \mathbf{e}_3\rangle}{\|\dot\gamma\|^2}\,dt \tag{6.2}$$

for any $t_0 \in \mathbb{R}$.

A second example is $F(t, s) = \gamma(t) + s\mathbf{a}$, showing that the index of γ does not change under translation of the curve.

6.2 Umlaufsatz

HOPF'S UMLAUFSATZ is for regular simple closed curves in the plane. Recall that a curve is simple of period ℓ if

$$\gamma(t) = \gamma(t') \qquad \text{if and only if} \qquad t = t' + n\ell \quad \text{for some } n \in \mathbb{Z}.$$

Theorem 6.5. *Let γ be a regular simple closed plane curve. Then*

$$\operatorname{ind}(\gamma) = \pm 1.$$

Proof. Since $\gamma(\mathbb{R}) = \gamma([0, \ell])$ is a compact subset of the plane, there is a t such that the y-component of $\mathbf{p} = \gamma(t)$ is minimal. Translate γ so that $\mathbf{p} = \mathbf{0}$ and reparameterise so that $\gamma(0) = \mathbf{0}$. Then $\dot\gamma(0)$ is parallel with $\mathbf{e}_1$ and γ lies in the half-plane above the x-axis.

Consider the function

$$\psi(t, s) = \begin{cases} \dfrac{\gamma(s) - \gamma(t)}{s - t}, & \text{for } s \neq t,\\[2ex] \dot\gamma(t), & \text{for } s = t. \end{cases}$$

This is continuous and smooth so there is a smooth determination $\Psi(t, s)$ of the angle between $\psi(t, s)$ and $\mathbf{e}_1$ given by using Lemma 6.1 in first one variable and then in the other variable. Along $s = t$ this determination of angle differs from that for $\dot\gamma$ by a constant integer multiple of 2π, so we have

$$2\pi \operatorname{ind}(\gamma) = \Psi(\ell, \ell) - \Psi(0, 0) = \big(\Psi(\ell, \ell) - \Psi(0, \ell)\big) + \big(\Psi(0, \ell) - \Psi(0, 0)\big).$$

Suppose $\Psi(0,0) = 0$. Then $\psi(0,s)$ lies in the upper half-plane for all s, so $\Psi(0,s) \in [0,\pi]$. As $s \to \ell$, $\psi(0,s)$ approaches $\mathbf{0}$ from the left, so $\Psi(0,\ell) = \pi$. On the other hand, $\Psi(t,\ell)$ lies only in the lower half-plane, and so $\Psi(t,\ell) \in [\pi, 2\pi]$ giving $\Psi(\ell,\ell) = 2\pi$, thus $\operatorname{ind}(\gamma) = (2\pi - 0)/\pi = +1$.

If $\Psi(0,0) = \pi$, then one finds the picture is reversed and $\operatorname{ind}(\gamma) = -1$. □

6.3 Relative Index

Given smooth orthonormal bases $\mathcal{B} = (\mathbf{e}', \mathbf{e}'')$ and a regular curve γ we may write $\dot\gamma / \|\dot\gamma\| = ce' + se''$ and use Lemma 6.1 to find a smooth function $\phi_{\mathcal{B}}$ with

$$\dot\gamma(t) = \|\dot\gamma(t)\| \left(\mathbf{e}' \cos\phi_{\mathcal{B}}(t) + \mathbf{e}'' \sin\phi_{\mathcal{B}}(t)\right).$$

If γ is a simple closed curve of period ℓ, we define the index of γ relative to $\mathcal{B}$ to be

$$\operatorname{ind}_{\mathcal{B}} \gamma = \frac{1}{2\pi}(\phi_{\mathcal{B}}(\ell) - \phi_{\mathcal{B}}(0)) \in \mathbb{Z}.$$

As before this index is isotopy invariant, and similar continuity arguments show that it is invariant under smooth changes of basis $\mathcal{B}$.

6.4 From surfaces to the plane

The situation we wish to consider is the following. Let $\sigma(U) \subset \mathcal{S}$ be a regular surface patch with U an open disc. Suppose γ is a regular, positively oriented, simple closed curve in $\sigma(U)$. This means that $\gamma(t) = \sigma(u(t), v(t))$ with $\pi(t) = (u(t), v(t))$ a regular simple closed curve in $U \subset \mathbb{R}^2$ such that the signed normal $\mathbf{n}_s$ points towards the interior $\operatorname{int}\pi$. Let $\mathcal{B} = (\mathbf{e}', \mathbf{e}'')$ be a smooth choice of positively oriented orthonormal basis for each $T_q\mathcal{S}$ for each $q \in \sigma(U)$. The relative index of the previous section may also be defined in this situation. To complete Pressley's proof of his Theorem 11.1 we need:

Theorem 6.6. $\operatorname{ind}_{\mathcal{B}} \gamma = +1$.

Proof. Choose a point $p \in \operatorname{int}\gamma = \sigma(\operatorname{int}\pi)$. We may assume that $p = \sigma(0,0)$. As $\mathcal{S}$ is a regular surface, we may relabel the axes in $\mathbb{R}^3$ and apply a translation so that there is a regular patch $\tilde\sigma(V) \subset \sigma(U)$ around p with

(i) $\tilde\sigma(x,y) = (x, y, f(x,y))$ with f smooth,

(ii) $p = \tilde\sigma(0,0)$, and

(iii) σ and $\tilde\sigma$ have the same orientation.

As $\sigma^{-1}(\tilde{\sigma}(V))$ is open in $U \subset \mathbb{R}^2$ there is an $\varepsilon > 0$ such that $\sigma(B_\varepsilon(0)) \subset \tilde{\sigma}(V)$.

Consider the family of curves $\gamma^\tau(t) = \sigma(\tau u(t), \tau v(t))$. We have that $\gamma_2 = \gamma^{\varepsilon/2}$ is isotopic to $\gamma = \gamma^1$ and that γ_2 lies in $\tilde{\sigma}(V)$. This means that

$$\gamma_2(t) = (x(t), y(t), f(x(t), y(t)))$$

for some smooth functions $x(t)$ and $y(t)$.

Now consider the curves $\gamma_2^\tau(t) = (x(t), y(t), \tau f(x(t), y(t)))$. We have $\gamma_2 = \gamma_2^1$ and $\tilde{\gamma} = \gamma_2^0$ is a simple positively oriented plane curve. Writing $\mathsf{e}' = (\mathsf{e}'_1, \mathsf{e}'_2, \mathsf{e}'_3)$, etc., consider also

$$\mathsf{f}'_\tau = (\mathsf{e}'_1, \mathsf{e}'_2, \tau \mathsf{e}'_3), \qquad \mathsf{f}''_\tau = (\mathsf{e}''_1, \mathsf{e}''_2, \tau \mathsf{e}''_3)$$

and the resulting orthonormal basis $\mathcal{B}_\tau$ given by the Gram-Schmidt process:

$$\mathsf{e}'_\tau = \mathsf{f}'_\tau / \left\|\mathsf{f}'_\tau\right\|, \qquad \mathsf{e}''_\tau = \frac{\mathsf{f}''_\tau - \langle \mathsf{f}''_\tau, \mathsf{e}'_\tau \rangle}{\left\|\mathsf{f}''_\tau - \langle \mathsf{f}''_\tau, \mathsf{e}'_\tau \rangle\right\|}.$$

These are smooth families so

$$\operatorname{ind}_{\mathcal{B}} \gamma = \operatorname{ind}_{\mathcal{B}} \gamma_2 = \operatorname{ind}_{\mathcal{B}_\tau} \gamma_2^\tau = \operatorname{ind}_{\tilde{\mathcal{B}}} \tilde{\gamma},$$

where $\tilde{\mathcal{B}} = \mathcal{B}_0$. Our problem is now reduce to a problem for the relative index in the plane.

The point $\mathsf{0} = (0, 0)$ lies in $\operatorname{int} \tilde{\gamma}$. Let ψ_0 be a choice of the angle between $\tilde{\mathsf{e}}'$ and e_1 at $\mathsf{0}$ so that $\tilde{\mathsf{e}}'_0 = (\cos \psi_0, \sin \psi_0)$. There is an $\tilde{\varepsilon} > 0$ such that $\left\langle \mathsf{e}'_q, \mathsf{e}'_0 \right\rangle > 1/2$ for all $|q| < \tilde{\varepsilon}$. Using the family $\tilde{\gamma}^\tau(t) = (\tau x(t), \tau y(t))$, we put $\hat{\gamma} = \tilde{\gamma}^{\tilde{\varepsilon}/2}$. This curve lies in $B_{\tilde{\varepsilon}}(\mathsf{0})$ and we have

$$\operatorname{ind}_{\mathcal{B}} \gamma = \operatorname{ind}_{\tilde{\mathcal{B}}} \tilde{\gamma} = \operatorname{ind}_{\tilde{\mathcal{B}}} \hat{\gamma}.$$

At each point of $B_{\tilde{\varepsilon}}(0)$ there is a unique choice of $\psi \in (\psi_0 - \pi/2, \psi_0 + \pi/2)$ so that $\tilde{\mathsf{e}}' = (\cos \psi, \sin \psi)$. The family of bases obtained from $\tilde{\mathsf{e}}'_\tau = (\cos(\tau\psi), \sin(\tau\psi))$ is smooth and at $\tau = 0$ is the standard basis for $\mathbb{R}^2$. We therefore have

$$\operatorname{ind}_{\mathcal{B}} \gamma = \operatorname{ind}_{\tilde{\mathcal{B}}} \hat{\gamma} = \operatorname{ind} \hat{\gamma}.$$

But $\operatorname{ind} \hat{\gamma} + 1$, by Hopf's Umlaufsatz, so the result follows. $\square$

References

[1] M. P. Do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, New Jersey, 1976.

[2] S. Newhouse, *Math 848 class notes: 10. Umlaufsatz*, `http://www.mth.msu.edu/~sen/Math_848-pdf/index.html`, August 2002.

[3] A. Pressley, *Elementary differential geometry*, Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., London, 2001.

Last revised: May 16, 2006.