Continuous functions on \( \mathbb{R} \)

N.J. Nielsen

November 5, 2008

We tacitly assume that the reader is familiar with the continuity properties of the classical functions defined on the \( \mathbb{R} \) or intervals of \( \mathbb{R} \). However, let us recall that if \( U \subseteq \mathbb{R} \) and \( f : U \to \mathbb{R} \) is a function, then \( f \) is said to be continuous in a point \( x_0 \in U \), if

\[
\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in U : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.
\] (1.1)

Intuitively speaking, this means that when \( x \) gets close to \( x_0 \), then \( f(x) \) gets close to \( f(x_0) \). \( f \) is said to be continuous if it is continuous in all points of \( U \). If we write this with quantifiers, we get:

\[
\forall \varepsilon > 0 \quad \forall x \in U \quad \exists \delta > 0 \quad \forall y \in U : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.
\] (1.2)

It is always a bit dangerous to interchange quantifiers in a logical statement because the statement changes radically. Let us anyway look on the following statement:

\[
\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in U \quad \forall y \in U : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.
\] (1.3)

If we do a little text analysis of the two statements we see that in (1.2) the \( \delta \) depends on \( \varepsilon \) and \( x \) while in (1.3) the \( \delta \) only depends on \( \varepsilon \) and thus works for all \( x, y \in X \). The statement (1.3) makes perfectly sense and gives rise to the following definition:

**Definition 1.1** Let \( U \subseteq \mathbb{R} \). A function \( f : U \to \mathbb{R} \) is called uniformly continuous if it satisfies

(1.3)

The word “uniformly” is used because given \( \varepsilon > 0 \), one can use the same \( \delta \) for all \( x, y \in X \). The next statement is really an example, but we formulate it as a proposition.

**Proposition 1.2** Let \( f : [1, \infty[ \to \mathbb{R} \) be defined by \( f(x) = \sqrt{x} \) for all \( 1 \leq x < \infty \). Then \( f \) is uniformly continuous.

**Proof:** Let \( x, y \geq 1 \) be arbitrary. Since \( f \) is differentiable, we can by the mean value theorem find a \( \xi \) between \( x \) and \( y \) so that

\[
f(x) - f(y) = f'(\xi)(x - y).
\]

Since \( \xi \geq 1 \) and \( f'(\xi) = \frac{1}{2\sqrt{\xi}} \), we get that \( |f'(\xi)| \leq \frac{1}{2} \) and hence

\[
|f(x) - f(y)| \leq \frac{1}{2}|x - y|
\]
which holds for all \( x, y \geq 1 \). If now \( \varepsilon > 0 \) is arbitrary, we can choose a \( 0 < \delta < 2\varepsilon \) and if \( |x - y| < \delta \), then by the above:

\[
|f(x) - f(y)| \leq \varepsilon.
\]

This shows that \( f \) is uniformly continuous. \( \square \)

We shall later prove that any continuous function defined on a closed and bounded interval of \( \mathbb{R} \) is uniformly continuous. Combining this with Proposition 1.2 we get that the square root function is in fact uniformly continuous on \([0, \infty]\). The next example shows that even very nice continuous functions need not be uniformly continuous.

**Example 1.3** let \( g: \mathbb{R} \to \mathbb{R} \) be defined by \( g(x) = x^2 \) for all \( x \in \mathbb{R} \). We claim that \( g \) is not uniformly continuous. It is clearly enough to prove that \( g \) is not uniformly continuous on \([0, \infty]\). To see this we put \( \varepsilon = 1 \) and let \( 0 < \delta \leq 1 \) be arbitrary. If \( x \geq 0 \), we get

\[
0 \leq g(x + \delta) - g(x) = (x + \delta)^2 - x^2 = (2x + \delta)\delta.
\]

For all \( x > \frac{1}{2}(\delta^{-1} - \delta) \) we get that

\[
g(x + \delta) - g(x) > 1
\]

which shows that \( g \) is not uniformly continuous.