Branch and Bound for TSP

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The Symmetric TSP

$$
\begin{array}{lll}
\min & \sum_{(i,j)\in E} d_{ij}x_{ij} & \\
\text{s.t.} & x(\delta(i)) = 2 & i \in \{1,2,\ldots,n\} \\
& \sum_{i,j\in Z} x_{ij} \leq |Z| - 1 & \emptyset \subset Z \subset V \\
& x_{ij} \in 0,1 & (i,j) \in E
\end{array}
$$

Bounds

- One way to identify a bound for the TSP is by relaxing constraints. This could be to allow subtours. This bound is although know to be rather weak.
- An alternative is the **1-tree relaxation**.

The 1-tree bound

- Identify a special vertex **1** (this can be any vertex of the graph).
- **1** and all edges incident with **1** are removed from G.
- For the remaining graph determine the minimum spanning tree T.
- Now the two smallest edges e_1 and e_2 incident with **1** are added to T producing T_1 (called a **1-tree**)

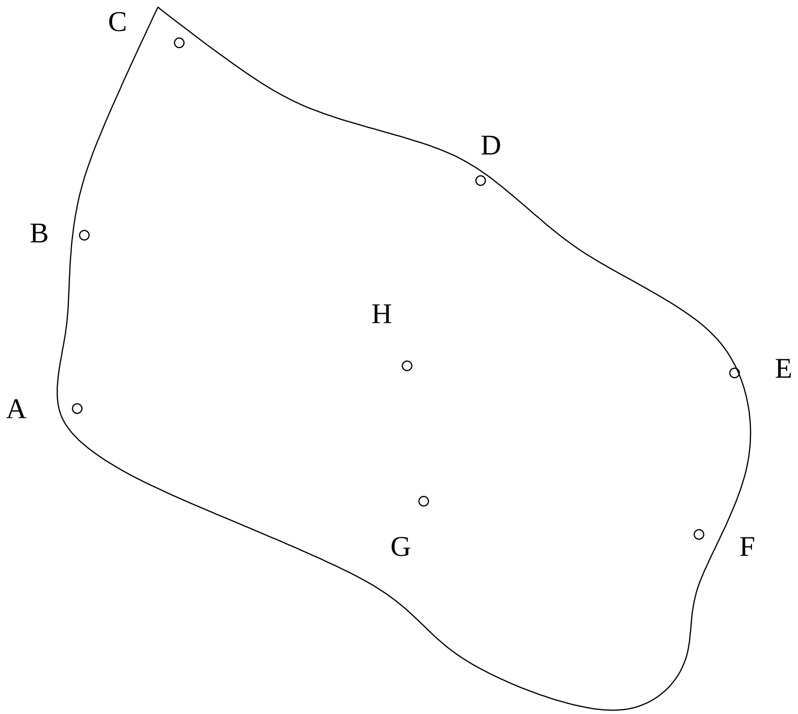
Why is T_1 a bound?

We need to convince ourselves that the total cost of T_1 is a lower bound of the value of an optimal tour.

- Note that a Hamiltonian tour can be divided into two edges e'_1 and e'_2 that are incident with **1** and the rest of the tour (let us call it T').
- So the set of Hamiltonian tours is a subset of 1-trees of G.
- Since e_1, e_2 are the two smallest edges incident to **1** $d_{e_1} + d_{e_2} \leq d_{e'_1} + d_{e'_2}$. Furthermore as T' is a tree $d(T) \leq d(T')$.

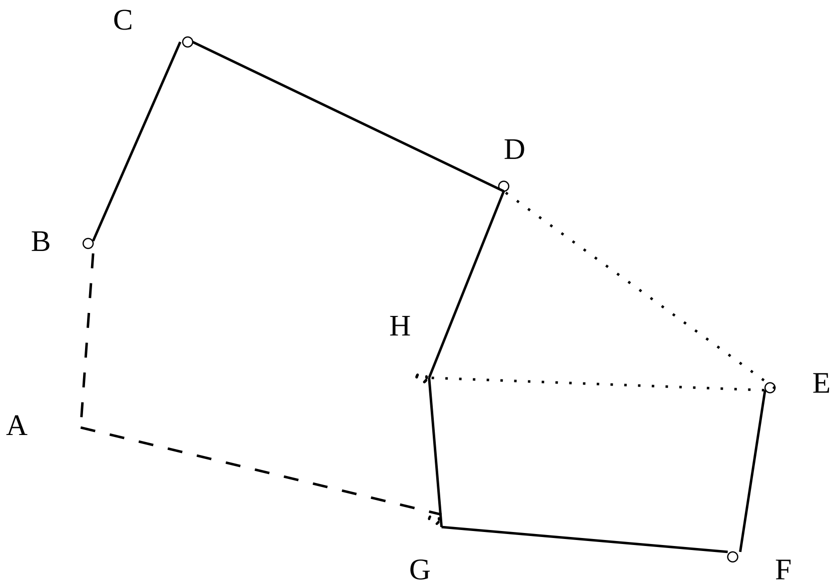
- So the cost of T_1 is less that or equal to the cost of any Hamiltonian tour.
- In the case T_1 is a tour we have found the optimal solution and can prune by bounding.
- otherwise we need to bound.

TSP of Bornbholm



	A	B	C	D	E	F	G	H
A	0	11	24	25	30	29	15	15
B	11	0	13	20	32	37	17	17
C	24	13	0	16	30	39	29	22
D	25	20	16	0	15	23	18	12
E	30	32	30	15	0	9	23	15
F	29	37	39	23	9	0	14	21
G	15	17	29	18	23	14	0	7
H	15	17	22	12	15	21	7	0

1-tree bound of Bornholm



Tree in rest of G

Edge left out by Kruskal's MST algorithm

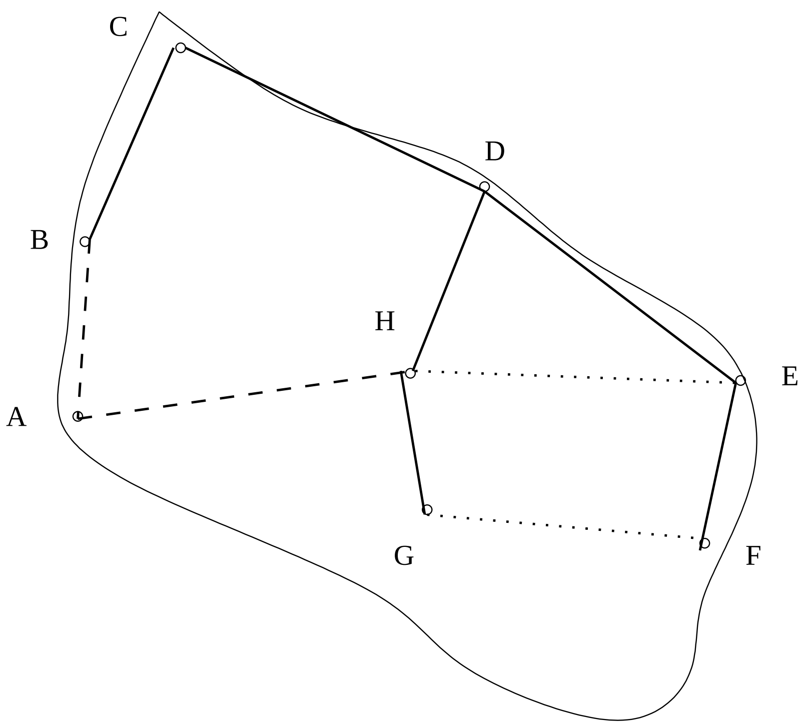
1−tree edge

Cost of 1−tree = 97

Strengthening the bound

- Idea: Vertices of T_1 with high degree are incident with too many attractive edges. Vertices of degree 1 have on the otherhand too many unattractive edges.
- Define π_i as the degree of vertex i minus 2.
- Note that $\sum_{i \in V} \pi_i$ equals 0 since T_1 has n edges and therefore the degree sum is $2n$.
- For each edge $(i, j) \in E$ we transform the cost to $c'_{ij} = c_{ij} + \pi_i + \pi_j$.

Strengthen the bound



Cost of 1−tree = 97

Modified distance matrix:

	A	B	C	D	E	F	G	H
A	0	11	24	25	29	29	16	15
B	11	0	13	20	31	37	18	17
C	24	13	0	16	29	39	30	22
D	25	20	16	0	14	23	19	12
E	29	31	29	14	0	8	23	14
F	29	37	39	23	8	0	15	21
G	16	18	30	19	23	15	0	8
H	15	17	22	12	14	21	8	0

How do we branch?

- Observe that in the case our 1-tree is **not** a tour at least one vertex has degree 3 or more.
- So choose a vertex v with degree 3 or more.
- For each edge (u_i, v) generate a subproblem where (u_i, v) is excluded from the set of edges.

Branching on Bornholm