Optimality and relaxation

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Optimality and Relaxation

Basic solution approach to any IP or COP:

$$z = \max\{c(x) : x \in X \subseteq Z^n\}$$

- Find lower bound (LB) $\underline{z}$ s.t. $\underline{z} \leq z$
- Find upper bound (UB) $\bar{z}$ s.t. $\bar{z} \geq z$

Now clearly $\bar{z} \geq \underline{z}$. Furthermore we have $\bar{z} = \underline{z} = z$ and we are done.

General approach

$$\underline{z}_1 < \underline{z}_2 < \underline{z}_3 < \ldots \leq z \leq \ldots \bar{z}_3 < \bar{z}_2 < \bar{z}_1$$

If $\bar{z}_t - \underline{z}_s < \epsilon$ then we may stop (and if $\bar{z}_t - \underline{z}_s = 0$ we have found an optimum.

Bounds

How do we actually find (upper and lower) bounds?

- **Primal bounds**: (lower bound for a max problem). Every feasible solution $x^* \in X$ is a lower bound.
- **Dual bounds**: (upper bound for a max problem). Most important approach is by "relaxation", that is, replace the original problem by an simpler optimization problem whose value is at least as large as z.

Relaxation

- A problem (RP) $z^R = \max\{f(x) : x \in T \subseteq R^n\}$ is a **relaxation** of (IP) $z = \max\{c(x) : x \in X \subseteq R^n\}$ if:
  - *(i)* $X \subseteq T$
  - *(ii)* for all $x \in X$: $c(x) \leq f(x)$

Relaxation

1. Linear Programming relaxation

2. Combinatorial relaxation

3. Lagrangian relaxation

Linear Programming relaxation

- For the integer program $\max\{cx : x \in P \cap Z^n\}$ with the formulation $P = \{x \in R^n_+ : Ax \leq b\}$ the **linear programming relaxation** is $\max\{cx : x \in P\}$.
- **Proporsition 2.3**:
  - ◇ *(i)* If a relaxation RP is infeasible, the original problem is infeasible.
  - ◇ *(ii)* Let x^* be an optimal solution to RP. If $x^* \in X$ and $f(x^*) = c(x^*)$ then x^* is an optimal solution to IP.

Combinatorial Relaxations

Whenever the relaxed problem is a combinatorial optimization problem we speak of a **combinatorial optimization**.

Lagrangian Relaxation

Consider an integer programming problem:

$$\begin{aligned} &\max cx \\ &Ax &\leq\ & b \\ &x &\in\ & X \end{aligned}$$

Now assume that if we dropped $Ax \leq b$ the problem

$$\begin{aligned} &\max cx \\ &x &\in\ & X \end{aligned}$$

would be "easy".

Now go one step further and add a penalty term (to the objective function) that is "active" when $Ax \leq b$ is violated, that is,

$$\begin{aligned} \max cx + u(b - Ax) & \\ x & \in X \end{aligned}$$

$z(u) = \max\{cx + u(b - Ax) : x \in X\}$ is called the **Lagrangian relaxation** of $z = \max\{cx : Ax \leq b, x \in X\}$.

Aside: How to spell Lagrangian

- Some spell it “Lagrangean”.
- Some spell it “Lagrangian”.
- Letting the one and only Google decide we get 467000 hits for Lagrangian and 12500 hits for Lagrangean. (12 Feb. 2004)
- So Lagrangian wins!!

Duality

- The two problems $z = \max\{c(x) : x \in X\}$ and $w = \min\{w(u) : u \in U\}$ form a **(weak-)dual pair** if $c(x) \leq w(u)$ for all $x \in X, u \in U$.
- If $z = w$, that is, there exists $x^* \in X$ and $u^* \in U$ s.t. $c(x^*) = w(u^*)$ they form a **(strong-)dual pair**.

Linear programming relaxations immediately leads to a weak dual.

- The integer program $z = \max\{c(x) : Ax \leq b, x \in Z^n_+\}$ and the linear program $w^{LP} = \min\{ub : uA \geq c, u \in R^m_+\}$ form a weak dual pair.
- Suppose that IP and D are a weak-dual pair.
  - (i) If D is unbounded, IP is infeasible.
  - (ii) If $x^* \in X$ and $u^* \in U$ satisfy $c(x^*) = w(u^*)$ then x^* is optimal for IP and u^* is optimal for D.

Greedy algorithms

- A greedy algorithm obtains an optimal solution to a problem by making a sequence of choices. For each decision point in the algorithm, the choice that seems best at the moment is chosen.
- This strategy does *not* always produce an optimal solution, but sometimes it does.