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Fractional colouring and density of sets avoiding distance 1

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Abstract:

A long-standing mathematical problem is to determine how big a set of points that does not contain two points at distance exactly one from each other can be. More precisely, given a normed space $(\mathbb{R}^n, \|\cdot\|)$, the problem is to determine the maximum density, denoted by $m_1(\mathbb{R}^n, \|\cdot\|)$, that can be achieved by a set avoiding distance 1. In joint work with Christine Bachoc, Philippe Moustrou, Arnaud Pêcher and Antoine Sédillot, we develop an approach based on graph theory and obtain new results that we present in this talk.

If the norm is such that the unit circle is a polytope that tiles the space by translation (we call those norms "parallelohedron norms"), it is not hard to prove that sets avoiding distance 1 can achieve the density of $\frac{1}{2^n}$. However, this is not the case of the Euclidean norm, which is the most studied case. In the Euclidean plane, the best known lower bound on $m_1(\mathbb{R}^2)$ is about 0.229; such a set was build by Croft in 1967. Upper bounds are much more technical to prove and new tools to study this problem are still developed frequently and involve techniques from many fields of mathematics. Prior to our work, the best upper bound on $m_1(\mathbb{R}^2)$ was of 0.258795 and was obtained by Keleti et al. in 2015 through an approach based on harmonic analysis of autocorrelation functions. A long-term objective would be to prove a conjecture by Erdös that states that $m_1(\mathbb{R}^2)$ in the Euclidean plane is strictly smaller than 1/4. This conjecture therefore implies that the sets avoiding distance 1 cannot achieve the same density with the Euclidean norm than with a parallelohedron norm. However, even in the case of a parallelohedron norm, it is not known if $m_1(\mathbb{R}^n)$ is exactly equal to $\frac{1}{2^n}$ (this equality is known as the Bachoc-Robins conjecture).

Another closely related problem, known as the Hadwiger-Neslon problem, is to determine how many colours are needed to colour all the points of the Euclidean plane in such a way that no two points at distance exactly 1 receive the same colour. Even if this problem has been studied since at least the early 1950s, the only result we had was that the chromatic number of the plane was somewhere between 4 and 7, until de Grey proved in April of this year that it was at least 5. The two problems are related in that they both study the same graph, called the unit-distance graph of the plane: it is the graph whose vertices are the points of the plane and whose edges are the pair of points at distance exactly 1. The Hadwiger-Nelson problem studies its chromatic number while our problem studies its independence ratio.

In this talk, we strengthen the connection between the two problems by noticing that m_1 of a given normed space is upper bounded by the inverse of its fractional chromatic number. The fractional chromatic number of the Euclidean plane had already been studied but the upper bound that the known results implied on $m_1(\mathbb{R}^2)$ were far from as good as the bound by Keleti et al. In this work, we develop new techniques to study the fractional chromatic number of a normed space and use them to improve the upper bound on m_1 of the Euclidean plane and the lower bound on its chromatic number, as well as to prove the Bachoc-Robins conjecture for all the parallelohedron norms in dimension 2 and for several families of norms in dimension n.