Sequent calculi based on derivations
Preservation of properties by fibring

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Abstract. Fibring is a meta-logical constructor that combines two given logics and produces a new one. In particular, the fibring of two sequent calculi is obtained by combining the languages of both calculi and taking all rules allowed in either calculus. By their own nature, proofs in the fibring have no relationship to proofs in the components, so that these are essentially different objects. In this paper, we propose a novel definition of fibring of two sequent calculi that takes the notion of derivation as primitive. Using this construction, we show that a proof in the fibring is essentially a finite set of proofs in the components structured in a meaningful way. We also use this novel definition to show that fibring preserves cut elimination and decidability.

1 Introduction

Combining logics in an important topic in applied logics [4, 10, 1] that raises interesting theoretical problems related to transference results. The objective to produce a new logic from two (or more) given logics by using a meta operator which is the combination mechanism. Specially of interest is to investigate whether the mechanism preserves the logical properties of the original logics. In general, sufficient conditions can be given for preservation.

Fibring, proposed by Gabbay in [8], is one of the most challenging ones. Fibring can be and has been investigated from a deductive point of view (mainly using Hilbert calculus [18], labelled deductive systems [13] and tableau systems [6, 2]) and also from a model-theoretic perspective (using either an algebraic approach or a modal-like semantics [9]). Assume that a signature is a family C of sets C_k of connectives of arity k for each natural number k. Given two families C' and C'', the signatures of the component logics, the fibring of the signatures is the family $C' \cup C''$ where $(C' \cup C'')_k = C'_k \cup C''_k$ for each k. Hence, a formula in the fibring can have a mixture of the connectives of each component logic. The fibring of two Hilbert calculi (tableau systems) whose rules are given in a schematic way is an Hilbert calculus (tableau system) whose set of rules is the union of the rules of the component logics. Semantic-wise, a model of the fibring is such that its reduct to the signature of each component should be a model for that component. Several transference results were obtained, namely for soundness and completeness [18], several guises of interpolation and semi-decidability.

A particular case of fibring is fusion of modal logics [11, 17]. The fusion of two uni-modal logics is a bi-modal logic. In this context, more transference results were obtained namely preservation of the finite model property and preservation of decidability via the finite model property.

It is evident that the work both on fibring and on fusion was more directed towards semantic issues or at least where semantics plays an important role. A confirmation is that, for instance, fibring of sequent calculi has not been considered. As a consequence, there are no preservation results related e.g. with the preservation of cut elimination.

The objective of the paper is to present a novel definition of fibring of sequent calculi in such a way that we can display the role of derivations in the components with the derivations on the fibring. That means that we present sequent calculi via a relation relating sequences of sequents with sequents. We compare our approach with the usual one where the sequent calculus is presented by rules (structural and specific). The approach of fibring sequent calculus via derivations can also be used for heterogeneous fibring, that is when we want to define the fibring of two calculi presented in a different way [5], say for instance a sequent calculus and a tableau system.

In Section 2, we define signatures, formulas and substitution. In Section 3, we present fibring of sequent calculi presented by rules. In Section 4, we present fibring of sequent calculi given by derivations using the concept of translation of formulas from the fibring to formulas of the component logics and show the equivalence if the two presentations of fibring. Examples are given for fusion of modal logics. Section 5 deals with properties that are preserved by fibring, namely cut elimination and decidability. Some concluding remarks are made in Section 6.

2 Background

We only consider propositional-based sequent calculi. The formulas of such calculi are generated from a family of connectives.

Definition 1. A *signature* C is a family of sets indexed by the natural numbers. The elements of each C_k are called *constructors* or *connectives* of arity k. We say that $C \subseteq C'$ if $C_k \subseteq C'_k$ for every $k \in \mathbb{N}$.

Definition 2. Let C be a signature and $\Xi = \{\xi_n : n \in \mathbb{N}\}$ be a countable set of meta-variables. The *language* $L(C, \Xi)$ is the free algebra over C generated by Ξ. The elements of $L(C, \Xi)$ are called *formulas*.

The elements of Ξ are schema variables that will allow the definition of schematic derivations. A derivation can be obtained from a schematic derivation by using a substitution.

Throughout this paper, Ξ will be a fixed set; for this reason, we will usually abbreviate $L(C, \Xi)$ to $L(C)$.

Definition 3. A *substitution* is a map $\sigma : \Xi \to L(C)$. Substitutions can be inductively extended to formulas and to sets of formulas: $\sigma(\gamma)$ is the formula where each $\xi \in \Xi$ is replaced by $\sigma(\xi)$; $\sigma(\Gamma) = \{\sigma(\gamma) : \gamma \in \Gamma\}$.

In particular, when $\sigma(\xi_n) \in \Xi$ for every n, we say that σ is a *renaming of variables.*

3 Fibring of sequent calculi via rules

Sequent calculi are traditionally specified by a set of rules that fall into one of two categories: structural rules and rules for the connectives. In this section, we look at sequent calculi in this way and define fibring of sequent calculi given by rules.

3.1 Sequent calculi given by rules

Definition 4. A *sequent* over a signature C is a pair $\langle \Delta_1, \Delta_2 \rangle$, denoted by $\Delta_1 \longrightarrow \Delta_2$, where Δ_1 (the *antecedent*) and Δ_2 (the *consequent*) are multi-sets of formulas in $L(C)$.

We denote by Seq_C the set of sequents over C.

Definition 5. A *rule* is a pair $\langle \{\theta_1, \ldots, \theta_n\}, \gamma \rangle$, indicated by

$$\frac{\theta_1 \quad \ldots \quad \theta_n}{\gamma},$$

where $\theta_1, \ldots, \theta_n$ (the premises) and γ (the conclusion) are sequents.

Definition 6. A *sequent calculus (given by rules)* is a pair $\mathcal{R} = \langle C, R \rangle$, where C is a signature and R is a set of rules including structural rules and specific rules (for the connectives).

– Structural rules: chosen among

$$\frac{\xi_1, \Delta_1 \longrightarrow \Delta_2 \quad \Delta_1 \longrightarrow \Delta_2, \xi_1}{\Delta_1 \longrightarrow \Delta_2}\ \mathsf{Cut}$$

$$\frac{\Delta_1 \longrightarrow \Delta_2}{\xi_1, \Delta_1 \longrightarrow \Delta_2}\ \mathsf{LW} \qquad \frac{\Delta_1 \longrightarrow \Delta_2}{\Delta_1 \longrightarrow \Delta_2, \xi_1}\ \mathsf{RW}$$

$$\frac{\Delta_1, \xi_1, \xi_1 \longrightarrow \Delta_2}{\Delta_1, \xi_1 \longrightarrow \Delta_2}\ \mathsf{LC} \qquad \frac{\Delta_1 \longrightarrow \xi_1, \xi_1, \Delta_2}{\Delta_1 \longrightarrow \xi_1, \Delta_2}\ \mathsf{RC}$$

– Left rules: the antecedent of the conclusion includes a formula $c(\varphi_1, \ldots, \varphi_n)$ for some n-ary connective c.
– Right rules: the consequent of the conclusion includes a formula $c(\varphi_1, \ldots, \varphi_n)$ for some n-ary connective c.

The fundamental notion is the notion of derivation: when do we say that a sequent s is (sequent-)derivable from Δ?

Definition 7. A *(rule-)derivation* of a sequent s from a set of sequents Δ in sequent calculus $\mathcal{R}$ is a finite sequence $\Delta_{1,1} \longrightarrow \Delta_{2,1} \ldots \Delta_{1,n} \longrightarrow \Delta_{2,n}$ of sequents such that the following conditions hold.

- $\Delta_{1,1} \longrightarrow \Delta_{2,1}$ is s;
- for each $i = 1, \ldots, n$, one of the following holds:
  - $\Delta_{1,i} \longrightarrow \Delta_{2,i}$ is an axiom (justified by Ax), that is, $\Delta_{1,i} \cap \Delta_{2,i} \neq \emptyset$;
  - $\Delta_{1,i} \longrightarrow \Delta_{2,i} \in \Delta$ (justified by Hyp);
  - there exist a rule $r = \langle \{\theta_1, \ldots, \theta_k\}, \gamma \rangle$ and a substitution σ such that $\Delta_{1,i} \longrightarrow \Delta_{2,i} = \sigma(\gamma)$ and, for each $j = 1, \ldots, k$, there is $i < i_j \leq n$ with $\sigma(\theta_j) = \Delta_{1,i_j} \longrightarrow \Delta_{2,i_j}$ (justified by $r, i_1, \ldots, i_k$).

If such a derivation exists, we say that s is *derivable* from Δ, denoted by $\Delta \vdash_{\mathcal{R}} s$. When Δ is empty, we write simply $\vdash_{\mathcal{R}} s$.

The following examples illustrate this definition.

Example 1. The sequent calculus for minimal logic M has as only connective the implication $\rightarrow$, with arity two, and as rules the five structural rules together with the following two rules for implication.

$$\frac{\Gamma \longrightarrow \Delta, \xi_1 \quad \xi_2, \Gamma \longrightarrow \Delta}{(\xi_1 \rightarrow \xi_2), \Gamma \longrightarrow \Delta}\ \mathsf{L}\rightarrow \qquad \frac{\xi_1, \Gamma \longrightarrow \Delta, \xi_2}{\Gamma \longrightarrow \Delta, (\xi_1 \rightarrow \xi_2)}\ \mathsf{R}\rightarrow$$

The following derivation ω_M shows that $\vdash_M \longrightarrow (\xi_1 \rightarrow (\xi_2 \rightarrow \xi_1))$.

1. $\longrightarrow (\xi_1 \rightarrow (\xi_2 \rightarrow \xi_1))$	$\mathsf{R}\rightarrow, 2$
2. $\xi_1 \longrightarrow (\xi_2 \rightarrow \xi_1)$	$\mathsf{R}\rightarrow, 3$
3. $\xi_2, \xi_1 \longrightarrow \xi_1$	Ax

Example 2. The sequent calculus for minimal logic with an $S4$ modality (characterized by Kripke structures with a transitive accessibility relation), which we will denote by $S4$, has two unary connectives $\Box$ and $\Diamond$, a binary connective $\rightarrow$ and as rules those of M together with the following four rules for the two modalities.

$$\frac{\Gamma, \xi_1, (\Box\xi_1) \longrightarrow \Delta}{\Gamma, (\Box\xi_1) \longrightarrow \Delta}\ \mathsf{L}\Box \qquad \frac{\Box\Gamma_1 \longrightarrow \xi_1, \Delta_1}{\Gamma_2, \Box(\Gamma_1) \longrightarrow (\Box\xi_1), \Diamond(\Delta_1), \Delta_2}\ \mathsf{R}\Box$$

$$\frac{\xi_1, \Gamma_1 \longrightarrow \Diamond(\Delta_1)}{(\Diamond\xi_1), \Box(\Gamma_1), \Gamma_2 \longrightarrow \Delta_2, \Diamond(\Delta_1)}\ \mathsf{L}\Diamond \qquad \frac{\Gamma \longrightarrow \Delta, \xi_1, (\Diamond\xi_1)}{\Gamma \longrightarrow \Delta, (\Diamond\xi_1)}\ \mathsf{R}\Diamond$$

In these rules, $\Box(\Gamma) = \{(\Box\varphi) : \varphi \in \Gamma\}$ (and similarly for $\Diamond(\Gamma)$).

The following derivation ω_N shows that $\{\longrightarrow \xi_1\} \vdash_{S4} \longrightarrow (\Box\xi_1)$.

1. $\longrightarrow (\Box\xi_1)$	$\mathsf{R}\Box, 2$
2. $\longrightarrow \xi_1$	Hyp

It is worth stressing that $\not\vdash_{S4}\ \xi_1 \longrightarrow (\Box\xi_1)$, so allowing hypotheses in the derivations is an essential feature of our definition – as is quite well-known by people working in modal logic.

Another interesting example of a derivation in this system is the following proof ω_{S4} of $\vdash_{S4} \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1)))$.

$$
\begin{array}{ll}
1.\ \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))) & \mathsf{R}\Diamond, 2 \\
2.\ \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\xi_1 \to (\Box\xi_1)) & \mathsf{R}\to, 3 \\
3.\ \xi_1 \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\Box\xi_1) & \mathsf{R}\Box, 4 \\
4.\ \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), \xi_1 & \mathsf{R}\Diamond, 5 \\
5.\ \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\xi_1 \to (\Box\xi_1)), \xi_1 & \mathsf{R}\to, 6 \\
6.\ \xi_1 \longrightarrow (\Diamond(\xi_1 \to (\Box\xi_1))), (\Box\xi_1), \xi_1 & \mathsf{Ax}
\end{array}
$$

Example 3. Finally we introduce a sequent calculus D for propositional logic with connectives $\neg$ and $\to$ together with a D modality (characterized by Kripke structures where every world can access another one). This calculus, see for instance [14], has as rules the five structural rules, the two rules $\mathsf{L}\to$ and $\mathsf{R}\to$ present in $S4$, left and right rules for $\neg$ and right rules for the modalities.

$$
\frac{\Gamma \longrightarrow \Delta, \xi_1}{\Gamma, (\neg\xi_1) \longrightarrow \Delta}\ \mathsf{L}\neg \qquad \frac{\Gamma, \xi_1 \longrightarrow \Delta}{\Gamma \longrightarrow (\neg\xi_1), \Delta}\ \mathsf{R}\neg
$$

$$
\frac{\Gamma \longrightarrow \xi_1}{\Box(\Gamma) \longrightarrow (\Box\xi_1)}\ \mathsf{R}\Box \qquad \frac{\Gamma \longrightarrow \xi_1}{\Box(\Gamma) \longrightarrow (\Diamond\xi_1)}\ \mathsf{R}\Diamond
$$

The following derivation ω_D shows that $\longrightarrow \xi_2 \vdash_D \longrightarrow (\Diamond(\xi_1 \to \xi_2))$.

$$
\begin{array}{ll}
1.\ \longrightarrow (\Diamond(\xi_1 \to \xi_2)) & \mathsf{Cut}, 2, 5 \\
2.\ (\Box\xi_2) \longrightarrow (\Diamond(\xi_1 \to \xi_2)) & \mathsf{R}\Diamond, 3 \\
3.\ \xi_2 \longrightarrow (\xi_1 \to \xi_2) & \mathsf{R}\to, 4 \\
4.\ \xi_2, \xi_1 \longrightarrow \xi_2 & \mathsf{Ax} \\
5.\ \longrightarrow (\Diamond(\xi_1 \to \xi_2)), (\Box\xi_2) & \mathsf{RW}, 6 \\
6.\ \longrightarrow (\Box\xi_2) & \mathsf{R}\Box, 7 \\
7.\ \longrightarrow \xi_2 & \mathsf{Hyp}
\end{array}
$$

We conclude this section with a small result on derivations.

Proposition 1. Let $\mathcal{R} = \langle C, R\rangle$ be a sequent calculus given by rules, $\Delta \subseteq \mathsf{Seq}_C$ and $s \in \mathsf{Seq}_C$ such that $\Delta \vdash_{\mathcal{R}} s$ with derivation ω, and $\sigma : \Xi \to L(C)$ be a substitution. Then $\sigma(\Delta) \vdash_{\mathcal{R}} \sigma(s)$ with derivation $\sigma(\omega)$.

Proof. Straightforward from the definition of derivation.

3.2 Fibring

Definition 8. Let $\mathcal{R}' = \langle C', R'\rangle$ and $\mathcal{R}'' = \langle C'', R''\rangle$ be sequent calculi. The *fibring* $\mathcal{R}' \uplus \mathcal{R}''$ of $\mathcal{R}'$ and $\mathcal{R}''$ is the sequent calculus $\langle C' \cup C'', R' \cup R''\rangle$.

As an example of this definition, we show how we can construct from the examples above a logic with a propositional negation $\neg$ and implication $\rightarrow$, an $S4$ modality $\Box'$ and a D modality $\Box''$.

Example 4. Consider the sequent calculi $S4$ and D presented above, where the modalities are renamed $\Box'$ and $\Diamond'$ (from $S4$) and $\Box''$ and $\Diamond''$ (from D). Their fibring is the calculus $S4 \uplus D$ whose rules are all the rules presented in Examples 2 and 3.

In this system we can derive $\longrightarrow (\Diamond''(\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))))$ as follows.

1.	$\longrightarrow \Diamond''(\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	$\mathsf{Cut}, 2, 5$
2.	$(\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))) \longrightarrow (\Diamond''(\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))))$	$\mathsf{R}\Diamond'', 3$
3.	$(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))) \longrightarrow (\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	$\mathsf{R} \rightarrow, 4$
4.	$\xi_2, (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))) \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	Ax
5.	$\longrightarrow (\Diamond''(\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))), (\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	$\mathsf{RW}, 6$
6.	$\longrightarrow (\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	$\mathsf{R}\Box'', 7$
7.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	$\mathsf{R}\Diamond', 8$
8.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\xi_1 \rightarrow (\Box'\xi_1))$	$\mathsf{R} \rightarrow, 9$
9.	$\xi_1 \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\Box'\xi_1)$	$\mathsf{R}\Box', 10$
10.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), \xi_1$	$\mathsf{R}\Diamond', 11$
11.	$\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\xi_1 \rightarrow (\Box'\xi_1)), \xi_1$	$\mathsf{R} \rightarrow, 12$
12.	$\xi_1 \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\Box'\xi_1), \xi_1$	Ax

Notice, however, that we profit little from the ability to make derivations in $S4$ and D. A close look at the derivation above shows that ω_D and ω_{S4} (see Examples 2 and 3) appear at steps 1–7 and 7–12, respectively; but there is no way to identify them from the derivation given here. The fact that ξ_1 from ω_D has been replaced by $(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$ only complicates matters further. So, morally, this derivation is a new derivation, independent of ω_D and ω_{S4}.

4 Fibring of sequent calculi via derivations

With the notion of fibring presented above, a derivation in the fibring does not keep track of the derivations in the components it possibly originates from. Therefore, we now propose a generalization of the notion of sequent calculus where the notion of *derivation* (rather than that of rule) is central. This will allow us to define fibring in such a way that derivations in the fibring are built from derivations in the components, and from a derivation in the fibring we can immediately extract the original derivations.

4.1 Sequent calculi given by derivations

Definition 9. A *sequent calculus given by derivations* is a pair $\mathcal{D} = \langle C, P\rangle$ where C is a signature and $P = \{P_\Delta : \Delta \in \wp_{\text{fin}}\mathsf{Seq}_C\}$ is a family of predicates $P_\Delta \subseteq \mathsf{Seq}_C^* \times \mathsf{Seq}_C$ such that the following conditions hold.

– Conclusion: if $P_\Delta(\omega, s)$ holds, then s is the first element in ω.
– Monotonicity: if $\Delta_1 \subseteq \Delta_2$, then $P_{\Delta_1} \subseteq P_{\Delta_2}$.
– Closure under substitution: if $P_\Delta(\omega, s)$ holds and σ is a substitution, then $P_{\sigma(\Delta)}(\sigma(\omega), \sigma(s))$ also holds.

Definition 10. Let $\Delta \subseteq \mathsf{Seq}_C$ and $s \in \mathsf{Seq}_C$. We say that s is *derivable* from Δ in sequent calculus $\mathcal{D}$, denoted $\Delta \vdash_{\mathcal{D}} s$, if there exist a sequence ω of sequents and a finite set $\Delta' \subseteq \Delta$ such that $P_{\Delta'}(\omega, s)$ holds.

The next result shows that this notion generalizes the previous notion of sequent calculus.

Proposition 2. Let $\mathcal{R} = \langle C, R \rangle$ be a sequent calculus given by rules and define $\mathcal{D}(\mathcal{R}) = \langle C, P \rangle$ where $P_\Delta(\omega, s)$ holds iff ω is a rule-derivation of s from Δ. Then $\mathcal{D}(\mathcal{R})$ is a sequent calculus given by derivations. Furthermore, $\Delta \vdash_{\mathcal{R}} s$ iff $\Delta \vdash_{\mathcal{D}(\mathcal{R})} s$.

Proof. By definition a sequence can only be a derivation of its first sequent; the monotonicity of P_Δ is immediate from the definition of derivation in $\mathcal{R}$, while the closure for substitution follows from Proposition 1, so $\mathcal{D}(\mathcal{R})$ is a sequent calculus given by derivations.

If Δ is finite the last equivalence is straightforward. Otherwise, let ω be a derivation of s from Δ and consider the set $\Delta' \subseteq \Delta$ of hypotheses occurring in ω; Δ' is finite and $P_{\Delta'}(\omega, s)$ holds, so $\Delta \vdash_{\mathcal{D}(\mathcal{R})} s$. Conversely, if $\Delta \vdash_{\mathcal{D}(\mathcal{R})} s$ then $P_{\Delta'}(\omega, s)$ holds for some ω and finite $\Delta' \subseteq \Delta$; but then $\Delta' \vdash_{\mathcal{R}} s$, from which immediately follows that $\Delta \vdash_{\mathcal{R}} s$.

The advantage of this definition is that it allows derivations that are not justifiable by the application of rules. This will be essential for the definition of fibring.

Formally, in the passage from $\mathcal{R}$ to $\mathcal{D}(\mathcal{R})$, one forgets the justifications (since derivations in $\mathcal{D}(\mathcal{R})$ are simply sequences of sequents). However, since the only way to generate derivations in $\mathcal{D}(\mathcal{R})$ is by producing them in $\mathcal{R}$, we will assume that the justifications are kept available. This will be clear in the examples below; it will also be used in the proof of Proposition 7.

4.2 Translations

The fibring of two signatures is simply the union of the two signatures. Therefore, formulas in the original calculi can be seen as formulas in the fibring. In order to derive formulas that contain connectives from both systems (i.e., "mixed" formulas) we need to be able to represent these in the components. This is done by a general mechanism of translation that takes advantage of the fact that the set of variables is infinite.

Definition 11. Let C and C' be signatures with $C \subseteq C'$ and $g : L(C') \to \mathbb{N}$ be an injection. The *translation* $\tau_g : L(C') \to L(C)$ is a map defined inductively as follows:

– $\tau_g(\xi_i) = \xi_{2i+1}$ for $\xi_i \in \Xi$;
– $\tau_g(c) = c$ for $c \in C_0$;
– $\tau_g(c(\gamma'_1, \ldots, \gamma'_k)) = c(\tau_g(\gamma'_1), \ldots, \tau_g(\gamma'_k))$ for $c \in C_k$ and $\gamma'_1, \ldots, \gamma'_k \in L(C')$;
– $\tau_g(c'(\gamma'_1, \ldots, \gamma'_k)) = \xi_{2g(c'(\gamma'_1,\ldots,\gamma'_k))}$ for $c' \in C'_k \setminus C_k$ and $\gamma'_1, \ldots, \gamma'_k \in L(C')$.

Notice that the index of a variable in $\tau(L(C'))$ indicates whether that variable is the image of a variable or of a formula starting with a connective in $C' \setminus C$.

The translation of a set of a formulas, a sequent or a sequent of sequents is defined in the natural way.

Definition 12. With C, C' and g as above, $\tau_g^{-1} : \Xi \to L(C')$ is the following substitution:

– $\tau_g^{-1}(\xi_{2i+1}) = \xi_i$;
– $\tau_g^{-1}(\xi_{2i}) = g^{-1}(i)$.

From this point on, we assume g is fixed and write simply τ and τ^{-1}. The following lemma justifies the notation τ^{-1}.

Lemma 1. If $C \subseteq C'$, then $\tau^{-1} \circ \tau = \mathrm{id}$ and $\tau \circ \tau^{-1} = \mathrm{id}$.

Proof. Straightforward by induction.

4.3 Fibring

Definition 13. Let $\mathcal{D}' = \langle C', P' \rangle$ and $\mathcal{D}'' = \langle C'', P'' \rangle$ be sequent calculi given by derivations. The fibring $\mathcal{D}' \uplus \mathcal{D}''$ is the sequent calculus $\langle C, P \rangle$, where $C = C' \cup C''$ and each P_Δ is inductively defined as follows.

– if $P'_{\tau'(\Delta)}(\tau'(\omega), \tau'(s))$ holds, then $P_\Delta(\omega, s)$ also holds;
– if $P''_{\tau''(\Delta)}(\tau''(\omega), \tau''(s))$ holds, then $P_\Delta(\omega, s)$ also holds;
– for finite $\Sigma = \{s_1, \ldots, s_k\} \subseteq \mathsf{Seq}_C$, if $P_\Delta(\omega_i, s_i)$ holds for $i = 1, \ldots, k$ and $P_\Sigma(\omega_s, s)$ holds, then $P_\Delta(\omega, s)$ holds, where ω is the sequence of sequents

$$\omega_s \cdot \omega_1 \cdot \ldots \cdot \omega_k.$$

In this definition, τ' and τ'' denote the translations of $L(C)$ to $L(C')$ and $L(C'')$.

Proposition 3. With the definitions above, $\mathcal{D}' \uplus \mathcal{D}''$ defined above is a sequent calculus given by derivations.

The intuition is as follows: a derivation in the fibring is either a derivation in one of the components (modulo translation) or recursively built from derivations using these derivations as justifications for the hypotheses used. In particular, in the case where $\mathcal{D}'$ and $\mathcal{D}''$ are induced from sequent calculi presented by rules, each justification Hyp occurring in ω_s should be interpreted as "postponing" the proof of s_i until the point where ω_i begins.

This definition preserves the derivations in the components, which are joined at a higher level by concatenation. This captures the essence of a proof in the fibring in a much clearer way than the previous definition: it consists of proofs in the components that are joined together by a cut-like mechanism. The following examples illustrate this situation.

Example 5. Consider the systems $\mathcal{D}(M)$ and $\mathcal{D}(S4)$ induced by the sequent calculi presented in Examples 1 and 2, as well as their fibring $\mathcal{D}(M) \uplus \mathcal{D}(S4)$ where the implications are kept distinct. Writing $\rightarrow'$ for the M-implication and $\rightarrow''$ and $\Box''$ for the $S4$ connectives, we can show in this system that $\vdash_{\mathcal{D}(M)\uplus\mathcal{D}(S4)} \longrightarrow \Box''(\xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1))$. The derivation $\sigma(\omega_N) \cdot \omega_M$ proofs this fact, where ω_M was defined in Example 1, ω_N in Example 2 and $\sigma(\xi_1) = (\xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1))$:

- As shown above, ω_M proves that $\vdash_M \longrightarrow (\xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1))$. By Proposition 1, $\tau'(\omega_M)$ shows that $\vdash_M \longrightarrow \tau'(\xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1))$, which means that $P'_\emptyset(\tau'(\omega_M), \tau'(\xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1)))$ by definition of $\mathcal{D}(M)$. This implies that $P_\emptyset(\omega_M, \xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1))$.
- Similarly, ω_N proves $\longrightarrow \xi_1 \vdash_{S4} \longrightarrow \Box''(\xi_1)$; since $\tau'' \circ \sigma$ is a substitution in $L(C'')$, also $\tau'' \circ \sigma(\omega_N)$ proves $\longrightarrow \tau'' \circ \sigma(\xi_1) \vdash_{S4} \longrightarrow \tau'' \circ \sigma(\Box''(\xi_1))$, and therefore $P''_{\{\longrightarrow \tau''\circ\sigma(\xi_1)\}}(\tau'' \circ \sigma(\omega_N), \longrightarrow \tau'' \circ \sigma(\Box''(\xi_1)))$ holds, from which we conclude that $P_{\{\longrightarrow \sigma(\xi_1)\}}(\sigma(\omega_N), \longrightarrow \sigma(\Box''(\xi_1)))$ holds. By definition of σ, the latter is simply $P_{\{\longrightarrow(\xi_1 \rightarrow'(\xi_2 \rightarrow' \xi_1))\}}(\sigma(\omega_N), \longrightarrow \Box''(\xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1)))$
- Finally, from the definition of P we conclude from the two previous points that $P_\emptyset(\sigma(\omega_N) \cdot \omega_M, \longrightarrow \Box''(\xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1)))$ holds.

The following is the derivation $\sigma(\omega_N) \cdot \omega_M$. The boxes are shown for clarity.

1. $\longrightarrow (\Box''(\xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1)))$	$\mathsf{R}\Box', 2$
2. $\longrightarrow (\xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1))$	Hyp
1. $\longrightarrow (\xi_1 \rightarrow' (\xi_2 \rightarrow' \xi_1))$	$\mathsf{R} \rightarrow', 2$
2. $\xi_1 \longrightarrow (\xi_2 \rightarrow' \xi_1)$	$\mathsf{R} \rightarrow', 3$
3. $\xi_2, \xi_1 \longrightarrow \xi_1$	Ax

Example 6. Consider now systems $S4$ and D from Examples 2 and 3, with the modalities renamed $\Box'$ and $\Box''$, respectively. Considering the induced systems $\mathcal{D}(S4)$ and $\mathcal{D}(D)$, we see that $\vdash_{\mathcal{D}(S4)\uplus\mathcal{D}(D)} \longrightarrow (\Diamond''(\xi_2 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))))$ with derivation $\sigma(\omega_D) \cdot \omega_{S4}$, where $\sigma(\xi_1) = \xi_1$ and $\sigma(\xi_2) = (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$.

1. $\longrightarrow (\Diamond''(\xi_1 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))))$	$\mathsf{Cut}, 2, 5$
2. $(\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))) \longrightarrow (\Diamond''(\xi_1 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))))$	$\mathsf{R}\Diamond'', 3$
3. $(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))) \longrightarrow (\xi_1 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	$\mathsf{R} \rightarrow, 4$
4. $(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), \xi_1 \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	Ax
5. $\longrightarrow (\Diamond''(\xi_1 \rightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))), (\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	$\mathsf{RW}, 6$
6. $\longrightarrow (\Box''(\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))))$	$\mathsf{R}\Box'', 7$
7. $\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	Hyp
1. $\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1)))$	$\mathsf{R}\Diamond', 2$
2. $\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\xi_1 \rightarrow (\Box'\xi_1))$	$\mathsf{R} \rightarrow, 3$
3. $\xi_1 \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\Box'\xi_1)$	$\mathsf{R}\Box', 4$
4. $\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), \xi_1$	$\mathsf{R}\Diamond', 5$
5. $\longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\xi_1 \rightarrow (\Box'\xi_1)), \xi_1$	$\mathsf{R} \rightarrow, 6$
6. $\xi_1 \longrightarrow (\Diamond'(\xi_1 \rightarrow (\Box'\xi_1))), (\Box'\xi_1), \xi_1$	Ax

A first look at this example does not show much difference with Example 4. However, this last derivation differs from the previous one significantly when one takes a closer look at its structure. Because we now take derivations as primitive, in order to build a derivation in the fibring we need to produce derivations in the components, and these can be recovered from the result. It is also very clear that part of the derivation is being done in system $\mathcal{D}(S4)$ while the other part is done in system $\mathcal{D}(D)$. This captures the intuition behind the fibring in a much better way than simply joining the rules of the two systems.

This definition of fibring is apparently more restrictive than the previous one, since it seems that we cannot apply rules of $\mathcal{D}'$ to formulas of $\mathcal{D}''$. However, we will show that we can still do this by seeing the formulas of $\mathcal{D}''$ as formulas of the fibring and translating these into $\mathcal{D}'$ via τ'. Although the resulting proof is slightly more complicated, the advantadge is that this translation step is clearly indicated in the derivation one obtains.

4.4 Equivalence

This section is devoted to proving the following equivalence result: both definitions of fibring presented are equivalent when one considers the set of derivable sequents.

Throughout this section, we assume fixed two sequent calculi $\mathcal{R}' = \langle C', R' \rangle$ and $\mathcal{R}'' = \langle C'', R'' \rangle$ given by rules, such that Cut, LW and RW are in $R' \cup R''$, and define:

- $\mathcal{D}' = \mathcal{D}(\mathcal{R}')$ and $\mathcal{D}'' = \mathcal{D}(\mathcal{R}'')$ are the sequent calculi given by derivations induced by $\mathcal{R}'$ and $\mathcal{R}''$;
- $\mathcal{R} = \mathcal{R}' \uplus \mathcal{R}''$ is the fibring of $\mathcal{R}'$ and $\mathcal{R}''$;
- $\mathcal{D} = \mathcal{D}' \uplus \mathcal{D}''$ is the fibring of $\mathcal{D}'$ and $\mathcal{D}''$;
- $C = C' \cup C''$ is the common signature of $\mathcal{R}$ and $\mathcal{D}$.

The goal is to show that $\mathcal{D}$ and $\mathcal{R}$ are equivalent systems, in the sense that $\Delta \vdash_{\mathcal{R}} s$ iff $\Delta \vdash_{\mathcal{D}} s$, for any $\Delta \subseteq \mathsf{Seq}_C$ and $s \in \mathsf{Seq}_C$. We begin by proving the converse implication, which is quite simple.

Proposition 4. If $\Delta \vdash_{\mathcal{D}} s$, then $\Delta \vdash_{\mathcal{R}} s$.

Proof. If $\Delta \vdash_{\mathcal{R}} s$, then $P_\Delta(\omega, s)$ holds for some sequence of sequents ω. The result is proved by induction on the proof of $P_\Delta(\omega, s)$. Base: suppose that $P'_{\tau'(\Delta)}(\tau'(\omega), \tau'(s))$ holds; then $\tau'(\Delta) \vdash_{\mathcal{D}'} \tau'(s)$. From Proposition 2 it follows that $\tau'(\Delta) \vdash_{\mathcal{R}'} \tau'(s)$ and, since sequent calculi are closed for renaming of variables, $\tau'^{-1}(\tau'(\Delta)) \vdash_{\mathcal{R}'} \tau'^{-1}(\tau'(s))$, which, by Lemma 1 means precisely that $\Delta \vdash_{\mathcal{R}'} s$. Since rules of $\mathcal{R}'$ are included in those of $\mathcal{R}$, it follows that $\Delta \vdash_{\mathcal{R}} s$. The other base case is analogous. Step: suppose that ω is built from $\omega_1, \ldots, \omega_k$ and ω_s as in Definition 13, that $P_\Delta(\omega_i, s_i)$ holds for $i = 1, \ldots, k$ and $P_\Sigma(\omega_s, s)$ holds. By induction hypothesis, $\Delta \vdash_{\mathcal{R}} s_i$ for each i and $\Sigma \vdash_{\mathcal{R}} s$, that is, there are derivations ω'_i of s_i from Δ and ω'_s of s from Σ. Replacing each occurrence of a hypothesis s_i in ω'_s by the corresponding derivation ω'_i one obtains a valid rule derivation of s from Δ, hence $\Delta \vdash_{\mathcal{R}} s$.

For the direct implication, we need an auxiliary result.

Lemma 2. If $\omega = \Delta_{1,1} \longrightarrow \Delta_{2,1} \ldots \Delta_{1,n} \longrightarrow \Delta_{2,n}$ proves that $\Delta \vdash_{\mathcal{R}} s$, then $\Delta \vdash_{\mathcal{R}} \Delta_{1,i} \longrightarrow \Delta_{2,i}$ for $i = 2, \ldots, n$ with a derivation of length smaller than n.

Proof. $\Delta_{1,i} \longrightarrow \Delta_{2,i} \ldots \Delta_{1,n} \longrightarrow \Delta_{2,n}$ is a derivation of $\Delta_{1,i} \longrightarrow \Delta_{2,i}$ from Δ.

Of course there are even shorter proofs in general, which can be obtained from the one given above by removing irrelevant steps, but for our purposes this optimization is unnecessary.

Proposition 5. If $\Delta \vdash_{\mathcal{R}} s$, then $\Delta \vdash_{\mathcal{D}} s$.

Proof. By induction on the length n of the derivation of s from Δ. Recall that rules in $\mathcal{R}$ are either rules of $\mathcal{R}'$ or of $\mathcal{R}''$. Base: $n = 1$. Then s is either an axiom or an element of Δ, in which case $\tau'(s)$ is either an axiom or an element of $\tau'(\Delta)$ and in either case it follows that $\tau'(\Delta) \vdash_{\mathcal{D}'} \tau'(s)$, from which $\Delta \vdash_{\mathcal{D}} s$. Step: suppose $n > 1$ and consider the justification of $\Delta_{1,1} \longrightarrow \Delta_{2,1}$. If this is Ax or Hyp then the reasoning above still applies, so without loss of generality assume the justification is a rule $r = \langle \{\theta_1, \ldots, \theta_k\}, \gamma \rangle \in R'$. Then there exist a substitution σ such that $s = \sigma(\gamma)$ and $\sigma(\theta_j) = \Delta_{1,i_j} \longrightarrow \Delta_{2,i_j}$, with each $i_j \in \{2, \ldots, n\}$. By Lemma 2, $\Delta \vdash_{\mathcal{R}} \sigma(\theta_j)$ with a derivation of length smaller than n, so the induction hypothesis applies and we can conclude that $\Delta \vdash_{\mathcal{D}} \sigma(\theta_j)$; that is, for each j there is a sequence $\omega_j \in \mathsf{Seq}^*_C$ such that $P_{\Delta'_j}(\omega_j, \sigma(\theta_j))$ holds for some finite $\Delta'_j \subseteq \Delta$. Define $\Delta' = \bigcup\{\Delta_j : j = 1, \ldots, k\}$; then Δ' is still a finite set with $\Delta' \subseteq \Delta$ and such that $P_{\Delta'}(\omega_j, \sigma(\theta_j))$ holds. On the other hand, the sequence ω_s defined as

$$\begin{array}{rll} 1. & \tau'(s) & \mathsf{r}\ 2, \ldots, k+1 \\ 2. & \tau'(\sigma(\theta_1)) & \mathsf{Hyp} \\ & \vdots & \\ k+1. & \tau'(\sigma(\theta_k)) & \mathsf{Hyp} \end{array}$$

is a derivation of $\tau'(s)$ from $\tau'(\sigma(\Theta)) = \{\tau'(\sigma(\theta_1)), \ldots, \tau'(\sigma(\theta_k))\}$ in $\mathcal{R}'$, hence $P'_{\tau'(\sigma(\Theta))}(\omega_s, \tau'(s))$ holds, and thus $P_{\sigma(\Theta)}(\tau'^{-1}(\omega_s), s)$ holds. Therefore $P'_{\Delta'}(\omega, s)$ holds with ω built from $\tau'^{-1}(\omega_s)$ and the ω_j as in Definition 13. Since $\Delta' \subseteq \Delta$, we conclude that $\Delta \vdash_{\mathcal{D}} s$.

5 Preservation results

In this section, we study two properties of sequent calculi that are preserved by fibring: cut elimination (Section 5.1) and decidability (Section 5.2).

5.1 Cut elimination

Cut elimination is a property that can be expressed most naturally in terms of a sequent calculus given by rules. However, in order to show that this property is preserved by fibring (i.e. that the fibring of two systems with cut elimination also has cut elimination) we will need to consider the systems seen as calculi given by derivation using the equivalence proved above.

Definition 14. A sequent calculus given by rules $\mathcal{R} = \langle C, R\rangle$ has cut elimination iff, for any $\Delta \subseteq \mathsf{Seq}_C$ and $s \in \mathsf{Seq}_C$, whenever $\Delta \vdash_{\mathcal{R}} s$ there is a derivation ω for $\Delta \vdash_{\mathcal{R}} s$ that does not use the cut rule.

Proposition 6. Let $\mathcal{R}'$ and $\mathcal{R}''$ be sequent calculi given by rules with cut elimination. Then, their fibring $\mathcal{R}$ also has cut elimination.

Proof. Define $\mathcal{D}' = \mathcal{D}(\mathcal{R}')$, $\mathcal{D}'' = \mathcal{D}(\mathcal{R}'')$ and $\mathcal{D} = \mathcal{D}' \uplus \mathcal{D}''$ and suppose that $\Delta \vdash_{\mathcal{R}} s$. By Proposition 5, $\Delta \vdash_{\mathcal{D}} s$, that is, $P_\Delta(\omega, s)$ holds for some derivation ω. We establish the thesis by induction on ω. Base: $P'_{\tau'(\Delta)}(\tau'(\omega'), \tau'(s))$ holds; then $\tau'(\omega)$ proves $\tau'(\Delta) \vdash_{\mathcal{R}'} \tau'(s)$ and, since $\mathcal{R}'$ is closed under substitution (Proposition 1), $\Delta \vdash_{\mathcal{R}'} s$. Since cut elimination holds in $\mathcal{R}'$, there is a cut-free derivation ω' of $\Delta \vdash_{\mathcal{R}'} s$, which also establishes $\Delta \vdash_{\mathcal{R}} s$. The case where $P''_{\tau''(\Delta)}(\tau''(\omega''), \tau''(s))$ holds is similar. Step: suppose that ω is $\omega^*\omega_1 \dots \omega_n$ with $P_\Delta(\omega_i, s_i)$ and $P_{\{s_1,\dots,s_n\}}(\omega^*, s)$. By induction hypothesis, there are cut-free derivations ${\omega'}^*$ and $\omega'_1, \dots, \omega'_n$ in $\mathcal{R}$ such that ${\omega'}^*$ proves $\{s_1, \dots, s_n\} \vdash_{\mathcal{R}} s$ and ω'_i proves $\Delta \vdash_{\mathcal{R}} s_i$ for $i = 1, \dots, n$. Replacing each justification Hyp in ${\omega'}^*$ (see Proposition 4) we obtain a cut-free derivation proving $\Delta \vdash_{\mathcal{R}} s$.

5.2 Decidability

A useful property of a sequent calculus is the ability to decide whether a given derivation does indeed prove a sequent from a set of hypotheses. In this section we discuss under which conditions it is reasonable to expect this to hold and how this property behaves under fibring.

We will assume that the reader is familiar with the basics of recursion theory; also, we will assume the Church–Turing postulate throughout and work with the following definition of recursive set.

Definition 15. An n-ary relation S (on sequents, sequences of sequents) is *recursive* iff there is an algorithm that, given n arguments $x_1, \dots, x_n$ of the appropriate type, returns 1 if $S(x_1, \dots, x_n)$ holds and 0 otherwise.

A set S is recursive iff the relation '$\lambda x.x \in S$' is recursive.

By "algorithm" we mean a deterministic sequence of instructions that terminates on any given input.

It is not reasonable to expect that P_Δ be decidable for every given Δ. In fact, if $\mathcal{D}(\mathcal{R})$ is a sequent calculus induced by a calculus given by rules, then

$$\omega \equiv \quad 1. \quad s \quad \mathsf{Hyp}$$

is a valid derivation of s from Δ iff $s \in \Delta$; therefore, $P_\Delta(\omega, s)$ holds iff $s \in \Delta$, and hence Δ must be a recursive set. This motivates the following definition.

Definition 16. A sequent calculus given by derivations $\mathcal{D} = \langle C, P\rangle$ is *decidable* iff, for every recursive set $\Delta \subseteq \mathsf{Seq}_C$, the relation P_Δ is recursive.

A sequent calculus given by rules $\mathcal{R}$ is decidable iff $\mathcal{D}(\mathcal{R})$ is decidable.

Proposition 7. Let $\mathcal{R}$ be a sequent calculus given by rules. Then $\mathcal{R}$ is decidable iff for every rule r the relation S_r is recursive, where S_r is the relation such that $S_r(s_1, \ldots, s_n, s)$ holds iff $\langle\{s_1, \ldots, s_n\}, s\rangle$ is an instance of r.

Proof. First suppose that $\mathcal{R}$ is decidable and let $r = \langle\{\theta_1, \ldots, \theta_n\}, \gamma\rangle$ be a rule with n premisses. Given $s_1, \ldots, s$ and s, define a derivation ω by

$$\begin{array}{rl} 1.\ s & \mathsf{r}\ 2, \ldots, n+1 \\ 2.\ s_1 & \mathsf{Hyp} \\ \vdots & \\ n+1.\ s_n & \mathsf{Hyp} \end{array}$$

Then $P_{\{s_1,\ldots,s_n\}}(\omega, s)$ holds iff $(\{s_1, \ldots, s_n\}, s)$ is an instance of r; since the set $\{s_1, \ldots, s_n\}$ is finite, it is recursive, therefore $P_{\{s_1,\ldots,s_n\}}$ is recursive and therefore so is S_r.

Conversely, suppose that all S_r are recursive and let Δ be a recursive set of sequents. In order to show that P_Δ is recursive, consider given ω of length n and s. The following algorithm then allows one to decide whether $P_\Delta(\omega, s)$ holds or not, where $(\omega)_k$ denotes the kth element of the sequence ω.

1. If $(\omega)_1$ is not s, output 0.
2. For $k = 1, \ldots, n$ do
   (a) If $(\omega)_k$ is justified with Ax and no formula in the antecedent of $(\omega)_k$ occurs in its consequent, output 0.
   (b) If $(\omega)_k$ is justified with Hyp and does not occur in Δ, output 0.
   (c) If $(\omega)_k$ is justified with $r, i_1, \ldots, i_m$ and $S_r(\{(\omega)_{i_1}, \ldots, (\omega)_{i_m}\}, (\omega)_k)$ does not hold, output 0.
3. If all the checks above succeed, output 1.

Notice that termination is guaranteed because Δ is recursive (case Hyp) and because S_r is recursive (case r).

This proposition immediately yields the following result.

Corollary 1. Let $\mathcal{R}'$ and $\mathcal{R}''$ be decidable sequent calculi given by rules. Then their fibring $\mathcal{R} = \mathcal{R}' \uplus \mathcal{R}''$ is decidable.

Proof. By the previous proposition, if $\mathcal{R}'$ and $\mathcal{R}''$ are decidable then the relations $R_{r'}$ and $R_{r''}$ are decidable for each rule r' of $\mathcal{R}'$ and r'' of $\mathcal{R}''$; but these are precisely the rules of $\mathcal{R}$, so $\mathcal{R}$ is decidable.

The following result, however, is quite more general.

Proposition 8. Let $\mathcal{D}'$ and $\mathcal{D}''$ be decidable sequent calculi given by derivations. Then their fibring $\mathcal{D} = \mathcal{D}' \uplus \mathcal{D}''$ is decidable.

Proof. Let Δ, ω and s be given and assume that ω has length n. We need to decide whether $P_\Delta(\omega, s)$ holds.

The only difficult part is deciding how to split ω in case it is made up of smaller derivations. Since we are only interested in decidability, we will simply consider *all* possible partitions of ω (since these are in finite number); in most concrete cases, however, the correct partition can easily be determined (see the examples above) and the algorithm will be much more efficient.

The algorithm is as follows.

- For each partition of ω do
  1. If the partition is singular, check whether $P'_{\tau'(\Delta)}(\tau'(\omega), \tau'(s))$ holds or $P''_{\tau''(\Delta)}(\tau''(\omega), \tau''(s))$ holds. If either is the case, output 1; otherwise move to the next partition.
  2. Otherwise, let ω^* be the first sequence in the partition and $\omega_1, \ldots, \omega_n$ the remaining ones. Let s_i denote $(\omega_i)_1$.
  3. For each $i = 1, \ldots, n$ check whether $P_\Delta(\omega_i, s_i)$ holds. If this is not the case, go on to the next partition.
  4. If the test above succeeded for all i, check whether $P_{\{s_1,\ldots,s_n\}}(\omega, s)$ holds. If this is the case, output 1.
- When no partitions of ω are left, output 0.

The algorithm above always terminates, since the recursive calls are always on shorter derivations. Correctness is guaranteed, in the case of compound derivations, because the only hypotheses that can be used in the first subderivation must be the conclusions of the other subderivations.

6 Concluding remarks

In order to capture the relationship between derivations in the fibring with the derivations in the component logics, we introduce a new notion of fibring sequent calculi. The notion involves a translation technique that allow us to map a formula of the fibring into a formula of each component. This new notion of fibring sequent calculi is compared with a more usual one in which the sequent calculi are presented by rules.

Using this new notion, we prove that fibring of sequent calculi preserves cut elimination. This notion also provides a more adequate framework to study decidability, and we define decidable sequent calculus and prove that the fibring of decidable sequent calculi is still decidable.

Natural extensions of the work are to consider fibring sequent calculi for display logics [3, 16, 7] and fibring of labelled sequent calculi (for instance, fibring sequents for labelled modal logic where the labels are either worlds [15] or truth-values [12]. Also of interest would be to extend the work to the context of logics with quantifiers.

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