The Quest for Optimal Sorting Networks: Efficient Generation of Two-Layer Prefixes

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***Abstract*—Previous work identifying depth-optimal n-channel sorting networks for $9 \leq n \leq 16$ is based on exploiting symmetries of the first two layers. However, the naive generate-and-test approach typically applied does not scale. This paper revisits the problem of generating two-layer prefixes modulo symmetries. An improved notion of symmetry is provided and a novel technique based on regular languages and graph isomorphism is shown to generate the set of non-symmetric representations. An empirical evaluation demonstrates that the new method outperforms the generate-and-test approach by orders of magnitude and easily scales until $n = 40$.**

I. Introduction

Sorting networks are a Computer Science classic. Based on a very simple model, their underlying theory is surprisingly deep and complex. The study of sorting networks has intrigued computer scientists since the middle 1950s. Informally, a sorting network is a comparator network that sorts all of its inputs. A comparator network is a network constructed from n channels that carry n input values from "left to right" through a sequence of comparators. A comparator is a component attached to a pair of channels such that the pair of values coming in from the left come out sorted on the right. Consecutive comparators can be viewed as a "parallel layer" if no two touch on the same channel. For an overview on sorting networks see for example, Knuth [3] or Parberry [7].

Ever since sorting networks were introduced, there has been a quest to find optimal sorting networks for particular small numbers of inputs: optimal depth networks (in the number of parallel layers), as well as optimal size (in the number of comparators). In this paper we focus on optimal depth sorting networks.

Even today, very little progress has been seen. Optimal depth sorting networks for $n \leq 8$ are given by Knuth (1973), Section 5.3.4 of [3], which also details specific sorting networks for $n \leq 16$ with the smallest depths known at the time. In 1991, Parberry [8] showed that the networks given by Knuth are optimal for $n = 9$ and $n = 10$. Parberry's result was obtained by implementing an exhaustive search with pruning based on symmetries in the first two layers of the comparator networks, and executing the algorithm on a supercomputer (consuming 200 hours of low priority computation).

In 2011, Morgenstern and Schneider [6] applied SAT solvers to search for optimal depth sorting networks, and were able to reproduce the known results for $n < 10$ with an acceptable runtime, but still required 21 days of computation for $n = 10$, shredding any hope to achieve reasonable runtimes for $n \geq 11$. Optimality for the cases $11 \leq n \leq 16$ is shown by Bundala and Závodný (2014) in [1], first by showing that $n = 11$ requires at least depth 8, and then by showing that $n = 13$ requires at least depth 9. Their results are obtained using a SAT solver, and are also based on identifying symmetries in the first two layers of the sorting networks.

Both Parberry [8] and then Bundala and Závodný [1] consider the following question: what is the smallest set S of two-layer network prefixes that need be considered in the search for minimal depth sorting networks? In particular, such that, if no element of S can be extended to a sorting network of depth d, then no depth d sorting network exists. The approach in [1] identified 212 two-layer network prefixes for $n = 13$; however, the calculation of this set required 32 minutes of computation, and this approach does not scale for larger values of n.

In this paper, we show how to generate the same set of 212 two-layer prefixes for $n = 13$ in "under a second" and, following ideas presented in [1], improve results such that only 117 relevant two-layer prefixes need to be considered. Our approach also scales well, i.e. we can compute the set of 34,486 relevant prefixes for $n = 30$ in "under a minute", and that of relevant prefixes for $n = 40$ in around two hours. Our main contribution here is to illustrate how focusing on concepts of regular languages, graph isomorphism, and symmetry breaking facilitates the efficient generation of all two-layer prefixes modulo isomorphism of the networks.

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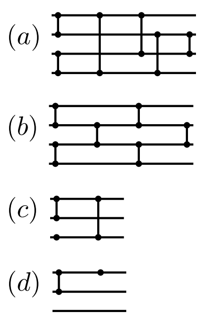
II. Preliminaries on Sorting Networks

A *comparator network* C with n channels and depth d is a sequence $C = L_1; \ldots ; L_d$ where each *layer* L_k is a set of comparators (i, j) for pairs of channels $i < j$. At each layer, every channel may occur in at most one comparator. A layer is *maximal* if it contains $\left\lfloor \frac{n}{2} \right\rfloor$ comparators. The *depth* of C is the number of layers d, and the *size* of C is the total number of comparators in its layers. If C_1 and C_2 are comparator networks, then $C_1; C_2$ denotes the comparator network obtained by concatenating the layers of C_1 and C_2; if C_1 has m layers, it is an *m-layer prefix* of $C_1; C_2$. An input $\bar{x} \in \{0, 1\}^n$ propagates through C as follows: $\bar{x}_0 = \bar{x}$, and for $0 < k \leq d$, $\bar{x}_k$ is a permutation of $\bar{x}_{k-1}$ obtained such that for each comparator $(i, j) \in L_k$, the values at positions i and j of $\bar{x}_{k-1}$ are reordered in $\bar{x}_k$ so that the value at position i is not larger than the value at position j. The output of the network for input $\bar{x}$ is $C(\bar{x}) = \bar{x}_d$, and $\mathsf{outputs}(C) = \left\{ C(\bar{x}) \mid \bar{x} \in \{0, 1\}^n \right\}$. The comparator network C is a *sorting network* if all elements of $\mathsf{outputs}(C)$ are sorted (in ascending order). The zero-one principle (e.g. [3]) implies that a sorting network also sorts any other totally ordered set, e.g. integers. The *optimal sorting network problem* is about finding the smallest depth and the smallest size of a sorting network for a given number of channels n.

A *generalized comparator network* is defined like a comparator network, except that it may contain comparators (i, j) with $i > j$, which order their outputs in descending order, instead of ascending. It is well known (Exercise 5.3.4.16 in [3]) that generalized sorting networks are no more powerful than sorting networks: a generalized sorting network can always be *untangled* into a (standard) sorting network with the same size and depth.



Images (a) and (b) on the right depict sorting networks on four channels, each consisting of four layers. The channels are indicated as horizontal lines (with channel 4 at the bottom), comparators are indicated as vertical lines connecting a pair of channels, and input values are assumed to propagate from left to right. Images (c) and (d) specify *patterns*. A pattern P is a partially specified network: it is a set of channels with comparators, but it may also include *external comparators*. These are singleton nodes representing a comparator connected to a channel not in P. A comparator network C contains a pattern P of depth d on m channels if there are a depth d prefix C_1 of C and a subset $\{c_1, \ldots , c_m\}$ of channels of C_1 such that: (i) $c_i < c_j$ if $i < j$; (ii) if P contains a comparator between channels i and j at layer $1 \leq k \leq d$, then C_1 contains a comparator between channels c_i and c_j at layer k; (iii) if P contains an external comparator touching channel i at layer $1 \leq k \leq d$, then C_1 contains a comparator between channel c_i and a channel $c \notin \{c_1, \ldots , c_m\}$ at layer k; (iv) C_1 contains no other comparators connecting to or between channels $c_1, \ldots , c_m$. The depth 2, three-channel pattern depicted in (c) occurs in network (a) but not in (b), while the pattern in (d) does not occur in either network (a) or (b): its third channel is never used, while all channels of (a) and (b) are used in the first two layers.

We can use permutations π on channels to manipulate (layers of) comparator networks. For a layer L, $\pi(L)$ contains the comparator $(\pi(i), \pi(j))$ iff L contains (i, j). If it is always the case that $\pi(i) < \pi(j)$, then $\pi(L)$ is also a layer, otherwise it is a *generalized layer*. The extension to networks is straightforward, and we write $C_1 \approx C_2$ (C_1 is equivalent to C_2) iff there is a permutation π such that C_1 is obtained by untangling the (generalized) comparator network $\pi(C_2)$. The two networks (a) and (b) above are equivalent via the permutation $(1\,3)(2\,4)$ and the application of the construction for untangling described in [3].

Parberry [8] shows that the first layer of a depth-optimal sorting network on n channels can be taken to consist of the comparators $(2k - 1, 2k)$ for $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. We denote this layer by F_n. The networks (a) and (b) have first layer F_4. In general, when $L_1; C$ is an n channel comparator network, we call a channel of C "min" ("max") if it is connected to the minimum (maximum) output of a comparator in L_1, and "free" if it does not occur in a comparator of L_1.

We make use of the following two lemmata, which are proved in [1]. The first lemma originates from [8].

Lemma 1. *Let π be a permutation such that $\pi(F_n) = F_n$ and let L be a layer on n channels such that $\pi(L)$ is a layer. If there is an n-channel sorting network of the form $F_n; L; C$ with depth d, then there is one of the form $F_n; \pi(L); C'$ with depth d.*

Lemma 2. *Let L_a and L_b be layers on n channels such that* $\mathsf{outputs}(F_n; L_b) \subseteq \pi(\mathsf{outputs}(F_n; L_a))$ *for some permutation π. If there is a sorting network $F_n; L_a; C$ of depth d, then there is also a sorting network $F_n; L_b; C'$ of depth d.*

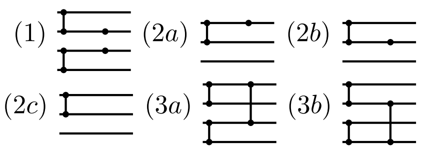
III. Saturation

Bundala and Závodný introduce, in [1], the notion of a saturated layer. We call a comparator network saturated if its last layer is saturated. The motivation is that, usually, adding a comparator to a network decreases the set of its possible outputs, but not always. When the network is saturated, adding a comparator to its last layer does not decrease the set of its possible outputs. This means that, when seeking a sufficient set of two layer networks with which to search for depth-optimal sorting networks, one can consider only saturated ones. The definition of saturation in [1] is syntactic. In this section, we propose a semantic characterization, and prove a syntactic criterion which is stronger than the one proposed therein. This means that we need to consider fewer two-layer networks.

Definition 1. *A comparator network C is* redundant *if there exists a network C' obtained from C by removing a comparator such that* $\mathsf{outputs}(C') = \mathsf{outputs}(C)$. *A network C is* saturated *if it is non-redundant and every network C' obtained by adding a comparator to the last layer of C satisfies* $\mathsf{outputs}(C') \not\subseteq \mathsf{outputs}(C)$.

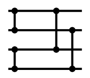
Parberry [8] shows that the first layer of a minimal-depth sorting network on n channels can always be assumed to contain $\lfloor \frac{n}{2} \rfloor$ comparators. Also, any comparator network that contains the same comparator at consecutive layers is redundant.

Theorem 1. *Let C be a saturated two-layer network. Then C contains none of the following two-layer patterns.*



Proof. Although this formulation is more general, the proof of case (1) is the same as the first case of the proof of Lemma 8 of [1], and the proof of cases $(2a)$, $(2b)$ and $(2c)$ is the same as the second case of the same proof.

For case $(3a)$, assume that C includes the given pattern and let the channels corresponding to those in the pattern be a, b, c and d. Add a comparator between channels b and d to obtain a network C' that includes the following pattern.



For a given input i of C', let i_k, m_k and o_k denote the values on channel k respectively at input, after layer 1, and at output. Exchanging the values of i_a with i_c and i_b with i_d has the effect of exchanging m_a with m_c and m_b with m_d. Then the output o can be obtained as an output of C:

- if $m_b \leq m_d$, then o is the output of C corresponding to input i;
- if $m_d < m_b$, then o is the output of C corresponding to input i' obtained from i by permuting i_a with i_c, i_b with i_d, and maintaining the value on all other channels.

Therefore C is not saturated.

For case $(3b)$ the construction is the same, and the thesis follows by comparing m_a with m_c. □

As it turns out, these are actually *all* of the patterns that make a comparator network with first layer F_n non-saturated. We formalize this observation in the following theorem.

Theorem 2. *If C is a non-redundant two-layer network on n channels with first layer F_n containing none of the patterns in Theorem 1, then C is saturated.*

Proof. Let C be a non-redundant two-layer comparator network, and assume that the second layer of C has at least two unused channels (otherwise there is nothing to prove). If one of these channels were unused at layer 1, then the network would contain the pattern $(2a)$, $(2b)$ or $(2c)$. Thus, by Theorem 1 necessarily the two channels are connected at layer 1. Again from the same theorem, we know that they must be both min-channels or both max-channels (otherwise case (1) applies) and the channels they are connected to at layer 1 cannot be connected at layer 2, otherwise the network would be redundant.

There are eight different cases to consider. We detail the cases where the two unused channels are max channels. Assume that the four relevant channels are adjacent. This does not lose generality, since a first-layer preserving permutation

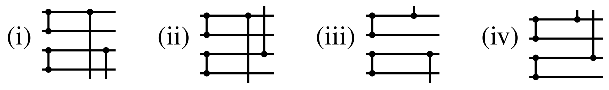


Fig. 1. Possible cases for channels a and c in the proof of Theorem 2. To obtain C', add a comparator between channels b and d.

can always be applied to C to make this hold. Label the channels a, b, c and d from top to bottom (so (a, b) and (c, d) are comparators at layer 1 and channels b and d are unused at layer 2). Let k be the number of channels above a and m be the number of channels below d. Adding a comparator to C yields C' where (b, d) is a comparator at layer 2. The four possibilities depend on whether channels a and c are min- or max-channels, and are represented in Figure 1.

- Figure 1 (i): a and c are min-channels at layer 2. Consider the input string $1^k 11001^m$. This is transformed to $1^k 10011^m$ by C', so $1^k 10011^m \in \mathsf{outputs}(C')$. We now show that $1^k 10011^m \notin \mathsf{outputs}(C)$. In order to obtain the 0 on channel b, the input string would necessarily have a 0 on channel a because of the comparator (a, b) at layer 1. But then the output would also have a 0 on channel a, hence it could not be $1^k 10011^m$.
- Figure 1 (ii): a is a min-channel at layer 2, and c is a max-channel.
The argument is similar, but using the input string $0^k 11001^m$. This is transformed to $0^k 10011^m$ by C', so $0^k 10011^m \in \mathsf{outputs}(C')$, and the same reasoning as above shows that $0^k 10011^m \notin \mathsf{outputs}(C)$.
- Figure 1 (iii): a is a max-channel at layer 2, and c is a min-channel.
Consider again the input string $1^k 11001^m$. As before, this is transformed to $1^k 10011^m$ by C', so $1^k 10011^m \in \mathsf{outputs}(C')$, and we show that $1^k 10011^m \notin \mathsf{outputs}(C)$. As before, to obtain the 0 on channel b the input string would necessarily have a 0 on channel a because of the comparator (a, b) at layer 1. Now this 0 is propagated upwards by the second-layer comparator at a, which means that the output has a 0 on one of the first k channels, hence it cannot be $1^k 10011^m$.
- Figure 1 (iv): a and c are both max-channels at layer 2. The reasoning is a bit more involved. Consider once more the input string $1^k 11001^m$. Since channel c is a second-layer max-channel connected w.l.o.g. to a channel $j \leq k$, the output produced by C' is $1^{j-1} 01^{k-j} 10111^m$. In order to obtain this output with network C, as before it is necessary to have inputs 0 on channels a and b; but since there are only two 0s in the output, this means that channel a must also be connected to channel j on layer 2, which is impossible.

The cases where a and c are the unused (min) channels are similar. □

We believe the following generalization to hold.

Conjecture 1. *If the two-layer networks C_1 and C_2 on n channels are both saturated and non-equivalent, then* $\mathsf{outputs}(C_1) \not\subseteq \mathsf{outputs}(C_2)$.

Particular cases of Conjecture 1 are implied by Theorem 2,

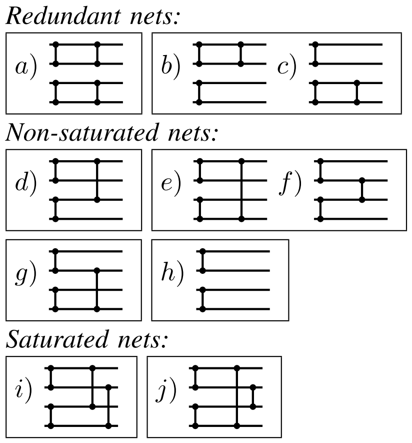


Fig. 2. The 10 two-layer standard networks on four channels with the Parberry first layer F_4.

but the general case remains open. The conjecture has been verified experimentally for $n \leq 15$.

IV. Case studies: $n = 4$ and $n = 5$

This section provides a detailed analysis for the cases of four-channel two-layer networks with first layer F_4 and five-channel two-layer networks with first layer F_5. Consider the following strategy to enumerate all possible second layers: channel 1 may be connected to channels 2, 3 or 4, or may be unused; if channel 1 is connected to channel 2, then channel 3 may be connected to channel 4 or may be unused; etc. With this strategy, the ten networks in Figure 2 are generated in the order $abidjefgch$.

The boxes around the networks represent classes of equivalent networks. There are only two non-trivial equivalence classes. The equivalence between nets $b)$ and $c)$ follows since the permutation $(1\,3)(2\,4)$ transforms them into one another. For nets $e)$ and $f)$, applying the same permutation to $e)$ yields a net that has a generalized comparator in layer 2; untangling it results in $f)$.

The nets in the first row are all redundant, as they repeat a comparator from the first layer; since the redundant comparators can be removed without altering the set of outputs, they can be simplified to net $h)$. The nets in the second row are not saturated; by Theorem 1, nets $d)$ and $g)$ are not saturated, and their extension $i)$ produces a subset of their outputs; a similar situation arises with net $e)$ vs net $j)$, and net $h)$ produces a superset of the outputs of both $i)$ and $j)$. We detail the sets of binary outputs for nets $d)$, $g)$ and $i)$, which correspond to Case $(3a)$ of Theorem 1, one of the two cases missing from the corresponding result in [1].

$$\begin{array}{rlllll}
\mathsf{outputs}(d) & = \{ & 0000, & 0001, & 0011, & 0100, \\
 & & 0101, & 0110, & 0111, & 1111 \quad \} \\
\mathsf{outputs}(g) & = \{ & 0000, & 0001, & 0011, & 0101, \\
 & & 0111, & 1001, & 1101, & 1111 \quad \} \\
\mathsf{outputs}(i) & = \{ & 0000, & 0001, & 0011, & 0101, \\
 & & 0111, & 1111 \quad \} & &
\end{array}$$

For $n = 5$ the situation is similar to $n = 4$. The generation

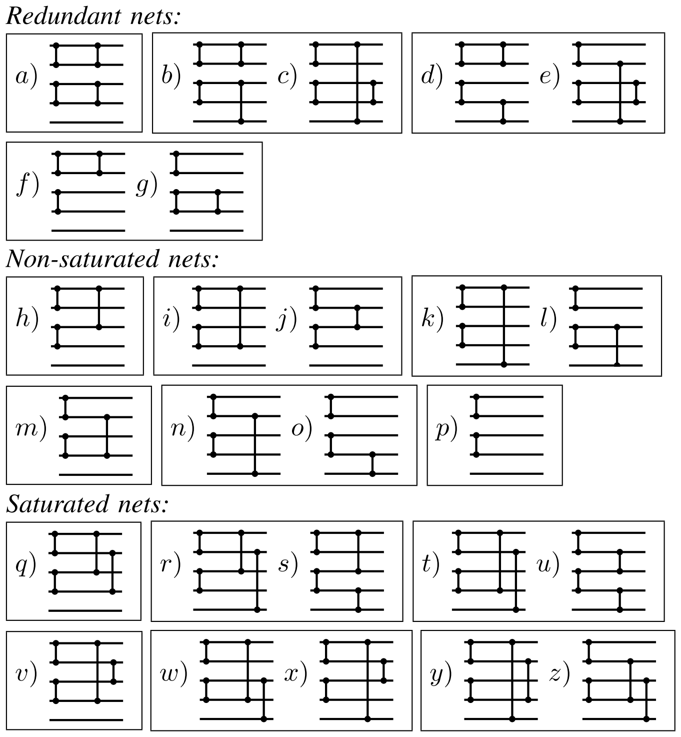


Fig. 3. The 26 two-layer standard networks on five channels with the Parberry first layer F_5.

algorithm from Section VI produces the 26 two-layer networks in Figure 3 in the order $abdfqrshvtwixyckujzmenglop$.

As before, the boxes identify the equivalence classes, which again can all be obtained by means of the permutation $(1\,3)(2\,4)$ and eventually reversing any generalized comparators at the second-layer. The first set of networks is redundant, while the second set is not saturated by Theorem 1, and once again it can easily be verified that each network in this group contains a set of outputs that is a proper superset of a network in the third group. Furthermore, only one element from each box in the third group needs to be considered.

Following the notation in [1], we denote the total number of two-layer networks on n channels whose first layer is F_n by $|G_n|$; the number of non-equivalent such networks (up to permutation of channels) by $|R(G_n)|$; and the corresponding values for saturated networks by $|S_n|$ and $|R(S_n)|$. From these analyses, we obtain $|G_4| = 10$, $|R(G_4)| = 8$, $|S_4| = |R(S_4)| = 2$; and $|G_5| = 26$, $|R(G_5)| = 16$, $|S_5| = 10$, and $|R(S_5)| = 6$. The values for $|G_4|$, $|R(G_4)|$, $|G_5|$ and $|S_5|$ coincide with those in [1], whereas the values we obtain for $|R(S_4)|$ and $|R(S_5)|$ coincide with those authors' results after applying Lemma 2 to eliminate representatives. The difference in values in $|R(G_5)|$ and $|R(S_5)|$ is probably due to an incomplete identification of the equivalence classes (note that, for $n = 5$, case (3) of Theorem 1 is not necessary, so the notion of saturated from [1] coincides with our definition in the previous section). The problem of computing the equivalence classes *efficiently* is the topic of the next sections.

V. Graph representation

The results presented in [1] involve a great deal of computational effort to identify permutations which render various two-layer networks equivalent. Motivated by the existence of sophisticated tools in the context of graph isomorphism, we adopt a representation for comparator networks similar to the one defined by Choi and Moon [2]. Let C be a comparator network on n channels. The graph representation of C is a directed and labeled graph, $\mathcal{G}(C) = (V, E)$ where each node in V corresponds to a comparator in C and $E \subseteq V \times \{min, max\} \times V$. Let $c(v)$ denote the comparator corresponding to a node v. Then, $(u, \ell, v) \in E$ if comparator $c(u)$ feeds into the comparator $c(v)$ in C and the label $\ell \in \{min, max\}$ indicates if the channel from $c(u)$ to $c(v)$ is the min or the max output of $c(u)$. Note that the number of channels cannot be inferred from the graph representation, as unused channels are not represented.

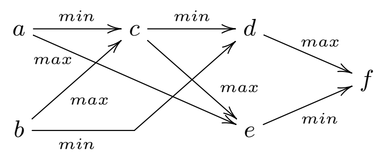
Each node has at most two in-edges and at most two out-edges. Nodes with less than two in-edges represent comparators that are connected to the input channels of the network. Similarly, nodes with less than two out-edges represent comparators which are connected to the output channels. As such, if the graph contains k comparators, then the sum of the in-degrees of the nodes and also the sum of the out-degrees of the nodes is bounded by $2k - n$.

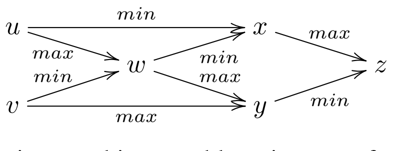
Clearly, graphs representing comparator networks are acyclic, and the degrees of their vertices are bounded by 4. There is a strong relationship between equivalence of comparator networks and isomorphism of their corresponding graphs. Choi and Moon [2] state the following proposition, which implies that the comparator network equivalence problem is polynomially reduced to the bounded-valence graph isomorphism problem.

Proposition 1. *Let C_1 and C_2 be n-channel comparator networks. Then*

$$C_1 \approx C_2 \Longleftrightarrow \mathcal{G}(C_1) \approx \mathcal{G}(C_2)\,.$$

Example 1. *The sorting networks (a) and (b) from Page 2 are represented by the following graphs, which can be seen to be isomorphic by mapping the vertices as $a \mapsto v$, $b \mapsto u$, $c \mapsto w$, $d \mapsto x$, $e \mapsto y$ and $f \mapsto z$.*





The graph isomorphism problem is one of a very small number of problems belonging to NP, for which it is neither known that they are solvable in polynomial time nor that they are NP-complete. However, it is known that the isomorphism of graphs of bounded valence (here: bounded degree) can be tested in polynomial time [4], so the comparator network equivalence problem can be efficiently solved.

An obvious approach for finding all two-layer prefixes modulo symmetry is to generate all two-layer networks as demonstrated in Section IV, and then apply graph isomorphism checking to find canonical representatives of the equivalence classes. We evaluated this approach using the popular graph isomorphism tool `nauty` [5], but found that the exponential growth in the number of two-layer prefixes prevents this approach from scaling.

Instead of a generate-and-test approach, in the next section we present a scalable method for directly generating only one representative two-layer prefix per equivalence class. Furthermore, this approach also enables us to encode saturation as a syntactic criterion in the generation process, i.e., to generate directly only representatives of saturated two-layer prefixes.

VI. Path representation of two-layer networks

In this section, we focus on two-layer networks where the first layer is maximal (although not necessarily F_n). These networks can be uniquely represented in terms of the paths in their graph representations. Furthermore, this representation can be read directly from the network, and can be used to construct a canonical representation of the network that completely characterizes the equivalence classes in the generated graphs. In the following, recall that channels of a network are characterized as *free*, *min* or *max* depending on the first layer.

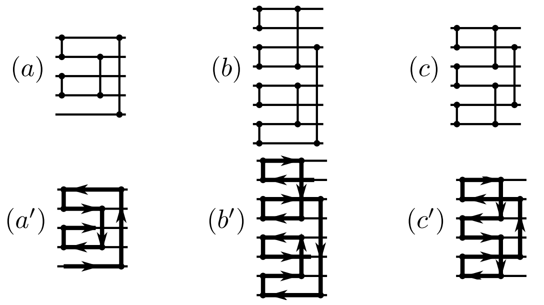
Definition 2. *A* path *in a two-layer network C is a sequence $\langle p_1 p_2 \dots p_k \rangle$ of distinct channels such that each pair of consecutive channels is connected by a comparator in C. The* word *corresponding to $\langle p_1 p_2 \dots p_k \rangle$ is $\langle w_1 w_2 \dots w_k \rangle$, where w_i is* 0, 1 *or* 2 *according to whether p_i is the free channel, a min channel or a max channel, respectively.*

A path is maximal if it is a simple path (with no repeated nodes) that cannot be extended (in either direction). A network is connected if its graph representation is connected.

Definition 3. *Let C be a connected two-layer network on n channels. Then* $\mathsf{word}(C)$ *is defined as follows.*

- Head *If n is odd, then $\mathsf{word}(C)$ is the word corresponding to the maximal path in C starting with the (unique) free channel.*
- Stick *If n is even and C has two channels not used in layer 2, then there are exactly two maximal paths in C starting with a free channel (which are reverse to one another), and $\mathsf{word}(C)$ is the lexicographically smallest of the words corresponding these two paths.*
- Cycle *If n is even and all channels are used by a comparator in layer 2, then $\mathsf{word}(C)$ is obtained by removing the last letter from the lexicographically smallest word corresponding to a maximal path in C that begins with two channels connected in layer 1.*

Example 2. *Below are three connected networks, (a), (b), and (c), with their maximal paths, pictured as (a′), (b′), and (c′), marked in bold. For instance, (a′) corresponds to the path* 51243.


Network (a) involves an odd number of channels, and the word corresponding to the maximal path (a') starting on the free channel is 01221*. Network (b) on an even number of channels contains two unused channels at layer 2, with two maximal paths (b') starting at a free channel and corresponding to the words* 21212112 *and* 21121212 *(its reverse); the corresponding word is thus the smallest of these two, namely* 21121212*. Finally, network (c) consists of a cycle, (c'). The words obtained by reading the possible maximal paths beginning with a layer 1 comparator are* 122121 *(starting on channel 3),* 121221 *and* 122112 *(starting on channels 1 and 5, respectively, and proceeding in the reverse direction). The lexicographically smallest of these is* 121221*, and thus the corresponding word is* 12122*.*

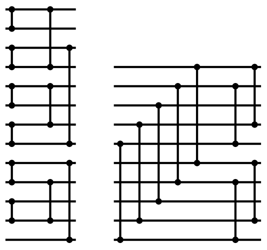
The set of all possible words (not necessarily minimal w.r.t. lexicographic ordering) can be described by the following BNF-style grammar.

$$\begin{aligned}\mathsf{Word} &::= \mathsf{Head} \mid \mathsf{Stick} \mid \mathsf{Cycle}\\ \mathsf{Head} &::= \mathsf{0}(\mathsf{12}+\mathsf{21})^*\\ \mathsf{Stick} &::= (\mathsf{1}+\mathsf{2})(\mathsf{12}+\mathsf{21})^*(\mathsf{1}+\mathsf{2})\\ \mathsf{Cycle} &::= \mathsf{12}(\mathsf{12}+\mathsf{21})^*(\mathsf{1}+\mathsf{2})\end{aligned}$$

Definition 4. *The* word representation *of a two-layer comparator network C,* $\mathsf{word}(C)$*, is the multi-set containing* $\mathsf{word}(C')$ *for each connected component C' of C; we will denote this set by the "sentence" $w_1; w_2; \dots; w_k$, where the words are in lexicographic order.*

In particular, a connected network will be represented by a sentence with only one word, so there is no ambiguity in the notation $\mathsf{word}(C)$. The restriction that layer 1 be maximal corresponds to the requirement that the multi-set $\mathsf{word}(C)$ have at most one Head-word.

Example 3. *The first network on the right consists of two connected components, which are nets (a) and (b) of Example 2. It is therefore represented by the sentence containing the words corresponding to those nets, namely* 01221; 21121212*.*



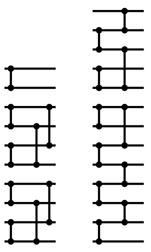
The second network consists of the first two layers of the 10-channel sorting network from Figure 49 of [3]. There are three connected components in this network, consisting of channels $\{1,4,6,9\}$, $\{2,5,7,10\}$ and $\{3,8\}$. The first two components contain similar cycles represented by the word 122*, while the third component yields the Stick-word* 12*. The whole network is thus represented by the sentence* 12; 122; 122*.*

Conversely, given a word $w = a_1 \dots a_\ell$ generated by the above grammar, we can generate a corresponding two-layer network $\mathsf{net}(w)$ as follows.

1) The number of channels n is: $|w|$, if $a_1 = \mathsf{0}$ or $|w|$ is even; and $|w|+1$, if $|w|$ is odd and $a_1 = \mathsf{1}$.
2) The first layer of $\mathsf{net}(w)$ is F_n.
3) If w is a Stick-word or a Cycle-word, ignore the first character; then, for $k = 0, \dots, \lfloor \frac{n}{2} \rfloor - 1$, take the next two characters xy of w and add a second-layer comparator between channels $2k + x$ and $2(k+1) + y$. If w is a Stick-word, ignore the last character; if w is a Cycle-word, connect the two remaining channels at the end.
4) If w is a Head-word, proceed as above but start by connecting the free channel to the channel indicated by the second character.

This construction can be adapted straightforwardly to obtain a net with any given first layer L_1: assuming the comparators in L_1 are numbered 1 to $\lfloor \frac{n}{2} \rfloor$, read $2k + x$ and $2(k+1) + y$ in step 3, as "the min/max channels from comparators k and $k+1$", where min or max is chosen according to x and y.

To generate a network from a sentence, simply generate the nets for each word in the sentence and compose them in the same order.



Example 4. *The two-layer networks on the right are generated from the sentences* 12; 122; 122 *and* 01221; 21121212*, respectively. It can readily be seen that these networks are equivalent to the ones in the previous example.*

Lemma 3. *Let C and C' be comparator networks on n channels. Then $C \approx C'$ iff* $\mathsf{word}(C) = \mathsf{word}(C')$*.*

Proof. The forward implication follows from the observation that, for two-layer networks, $C \approx C'$ means that there is a permutation π such that C' is $\pi(C)$ possibly with some generalized comparators in layer 2. Then, any path obtained in C beginning at channel j can be obtained in C' by beginning at channel $\pi(j)$, and reciprocally. The converse implication is straightforward. □

In algebraic terms, the function word can be seen as a "forgetful" functor that forgets the specific order of channels in a net, whereas net generates the "free" net from a given word. Furthermore, word always returns the minimum element in the "fiber" $\mathsf{net}^{-1}(w)$, whence lexicographically minimal words can be used to characterize equivalent nets. This means that word and net form an adjunction between suitably defined pre-orders.

As a consequence, the sets of all distinct two-layer networks on n channels, G_n, and their equivalence classes modulo permutations, $R(G_n)$, can be generated simply by generating all multi-sets of words with at most one Head-word yielding exactly n channels. This procedure has been implemented straightforwardly in Prolog, yielding the values in the table of Figure 4. Besides the values given in the table,

$|R(G_{20})| = 15{,}906$ was computed in a few seconds, and $|R(G_{30})| = 1{,}248{,}696$ in under a minute.

The sequence $|G_n|$ is actually known in Mathematics: it is sequence A000085 in The On-Line Encyclopedia of Integer Sequences,[1] and corresponds (among others) to the number of self-inverse permutations on n letters. The first two elements of the sequences coincide. Thus, to prove the above claim, it suffices to show that $|G_n|$ satisfies the characteristic recurrence for that sequence.

Theorem 3. $|G_n| = |G_{n-1}| + (n-1)\,|G_{n-2}|$ *for* $n \geq 3$.

Proof. The correspondence is at the level of layers, not of networks. Consider the following two operations on layers.

1) Given a layer L, $L^\bullet$ is L with an extra unused channel at the end.
2) Given a layer L, $L^{k\bullet}$ is L with two extra channels connected by a comparator: one between channels k and $k+1$ of L, the other at the end.

Given a layer L' on n channels, there is a unique way to write L' as $L^\bullet$ or $L^{k\bullet}$ (according to whether the last channel of L' is used), establishing the desired relationship. There seems to be no obvious relationship between the two layer networks containing L' and L as their second layers. □

Alternatively to considering the recurrence, one could argue that $|G_n|$ corresponds to the number of matchings in a complete graph with n nodes, since every comparator joins two channels.[2]

The sequence $|R(G_n)|$ does not appear to be known already, and it does not have such a simple description. The following properties are however interesting.

Theorem 4.

1) The number of non-equivalent redundant two-layer networks using n channels is $|R(G_{n-2})|$.
2) For odd n, $|R(G_n)| = |R(G_{n-1})| + 2\,|R(G_{n-2})|$.

Proof. The proof is based on the word representation of the nets.

1) If C is a redundant net, then the sentence $\mathsf{word}(C)$ contains 12. Removing one occurrence of this word yields a sentence corresponding to a network with $n-2$ channels. This construction is reversible, so there are $|R(G_{n-2})|$ sentences corresponding to redundant networks on n channels.
2) If n is odd, then $\mathsf{word}(C)$ contains exactly one word beginning with 0. If this word is 0, then removing it yields a network with $n-1$ channels, and this construction is reversible. Otherwise, removing the two last letters in this word yields a network with $n-2$ channels; since the removed letters can be 12 or 21, this matches each network on $n-2$ channels to two networks on n channels. □

The construction we described does not take into account the notion of saturation. However, the characterization of saturation given by Theorem 2 is straightforward to translate in terms of the word associated with a network.

Corollary 1. *Let C be a two-layer network. Then C is saturated if $w = \mathsf{word}(C)$ satisfies the following properties.*

1) If w contains 0 or 12, then all other words in sentence w are cycles.
2) No stick in w has length 4.
3) Every stick in w begins and ends with the same symbol.
4) If w contains a head or stick ending with c, then every head or stick in w ends with c, for $c \in \{1, 2\}$.

Thus, the set of saturated two-layer networks can be generated by using the following restricted grammar.

$$\begin{aligned}
\mathsf{Word} &::= \mathsf{Head} \mid \mathsf{Stick} \mid \mathsf{Cycle}\\
\mathsf{Stick} &::= \mathsf{12} \mid \mathsf{eStick} \mid \mathsf{oStick}\\
\mathsf{Head} &::= \mathsf{0} \mid \mathsf{eHead} \mid \mathsf{oHead}\\
\mathsf{eStick} &::= \mathsf{12}(\mathsf{12}+\mathsf{21})^{+}\mathsf{21}\\
\mathsf{eHead} &::= \mathsf{0}(\mathsf{12}+\mathsf{21})^{*}\mathsf{12}\\
\mathsf{oStick} &::= \mathsf{21}(\mathsf{12}+\mathsf{21})^{+}\mathsf{12}\\
\mathsf{oHead} &::= \mathsf{0}(\mathsf{12}+\mathsf{21})^{*}\mathsf{21}\\
\mathsf{Cycle} &::= \mathsf{12}(\mathsf{12}+\mathsf{21})^{*}(\mathsf{1}+\mathsf{2})
\end{aligned}$$

Furthermore, sentences are multi-sets M such that: (i) if M contains the words 0 or 12, then all other elements of M are cycles; (ii) if M contains an eHead or eStick, then it contains no oHead or oStick. With these restrictions, generating all saturated networks for $n \leq 20$ can be done almost instantaneously. The numbers S_n of saturated two-layer networks and $R(S_n)$ of equivalence classes modulo permutation are given in the first four lines in the table of Figure 4.

Bundala and Závodný mention that the number of two-layer networks could further be restricted by considering reflections [1] (with acknowledgement to D.E. Knuth). The reflection of a comparator network on n channels is the network obtained by replacing each comparator (i,j) by the comparator $(n-j+1, n-i+1)$; when the first layer is the set F'_n of comparators of the form $(i, n-i+1)$, reflection leaves it unchanged. Furthermore, they show that a two-layer network with first layer F'_n can be extended to a sorting network if, and only if, its reflection can be extended to a sorting network, hence reflections can be removed from $R(S_n)$ when searching for optimal depth sorting networks.

Since the word representation is defined for any first layer, this symmetry break can be encoded by a similar technique as the one applied for saturation. By removing eStick and eHead from the above grammar, we directly generate only the 118 representatives for 13-channel networks described in [1]. Furthermore, it is possible to have distinct cycles whose reflections are equivalent (but not equal); this brings the number of relevant two-layer networks on 13 channels to 117. The last line in the table of Figure 4 above details the number $|R_n|$ of representatives modulo equivalence and reflection for each value of $n \leq 40$. We can compute the set R_{30} in less than one minute and R_{40} in approximately two hours.

[1]https://oeis.org/A000085

[2]Thanks to Daniel Bundala for pointing out this observation.

n	3	4	5	6	7	8	9	10	11	12	13	14
$\|G_n\|$	4	10	26	76	232	764	2,620	9,496	35,696	140,152	568,504	2,390,480
$\|S_n\|$	2	4	10	28	70	230	676	2,456	7,916	31,374	109,856	467,716
$\|R(G_n)\|$	4	8	16	20	52	61	165	152	482	414	1,378	1,024
$\|R(S_n)\|$	2	2	6	6	14	15	37	27	88	70	212	136
$\|R_n\|$	1	2	4	5	8	12	22	21	48	50	117	94

n	15	16	17	18	19
$\|G_n\|$	10,349,536	46,206,736	211,799,312	997,313,824	4,809,701,440
$\|S_n\|$	1,759,422	7,968,204	31,922,840	152,664,200	646,888,154
$\|R(G_n)\|$	3,780	2,627	10,187	6,422	26,796
$\|R(S_n)\|$	494	323	1,149	651	2,632
$\|R_n\|$	262	211	609	411	1,367

n	20	21	22	23	24	25	26	27	28	29	30	31
$\|R(S_n)\|$	1,478	5,988	3,040	13,514	6,744	30,312	14,036	67,638	30,552	150,128	64,168	331,970
$\|R_n\|$	894	3,098	1,787	6,920	3,848	15,469	7,830	34,318	16,690	75,979	34,486	167,472

n	32	33	34	35	36	37	38	39	40
$\|R(S_n)\|$	138,122	731,000	291,090	1,604,790	622,136	3,511,250	1,313,262	7,663,112	2,792,966
$\|R_n\|$	73,191	368,143	152,503	806,710	322,891	1,763,133	676,431	3,843,848	1,429,836

Fig. 4. Table detailing the number of all distinct two-layer networks on n channels, G_n, the number of saturated such networks, S_n, the number of equivalence classes modulo permutations, $R(G_n)$, the number of saturated equivalence classes modulo permutations, $R(S_n)$, and the number of saturated equivalence classes modulo permutations and reflections, R_n. When searching for optimal-depth sorting networks, only networks extending R_n need to be considered.

Having computed R_{16}, we can directly verify the known value 9 for the optimal depth of a 16-channel sorting network, obtained only indirectly in [1]. This direct proof involves showing that none of the 211 two-layer comparator networks in R_{16} extends to a sorting network of depth 8. For this, we use an encoding to Boolean satisfiablity (SAT) as described in [1], where for each network C in R_{16}, we generate a formula φ_C that is satisfiable if and only if there exists a sorting network of depth 8 extending C. Showing the unsatisfiability of these 211 SAT instances can be performed in parallel, with the hardest instance (a CNF with approx. 450,000 clauses) requiring approx. 1800 seconds running on a single thread of a cluster of Intel Xeon E5-2620 nodes clocked at 2 GHz.

However, this approach does not directly work for $n = 17$, where the best known upper bound is 11. Attempting to show that there is no sorting network of depth 10 requires analyzing the networks in R_{17}. The resulting 609 formulas have more than five million clauses each, and none could be solved within a couple of weeks. It appears that finding the optimal depth of sorting networks with more than 16 channels is a hard challenge that will require prefixes with more than 2 layers.

VII. Conclusion

We presented an efficient technique to generate, modulo symmetry, the set $R(S_n)$ of all saturated two-layer comparator networks on n channels, as well as its restriction R_n to exclude networks that are equivalent modulo reflection.

As noted by Parberry in 1991 and again by Bundala and Závodný in 2014, computing $R(S_n)$ and R_n is a crucial step in the search for optimal depth sorting networks on n channels. Using our approach we can compute $R(S_{13})$ in under a second vs 30 minutes using the brute force approach applied in [1], and improve the number of relevant two-layer prefixes to be considered from 212 to 117 by eliminating reflections.

In personal communication, Bundala and Závodny state that their brute-force approach does not scale beyond $n = 13$. This is not suprising, as there is an exponential growth of the number of candidate networks, a quadratic number of subsumption tests between the candidate networks, and, for each subsumption test, a factorial number of permutations and an exponential number of inputs to consider.

The smallest open instance of the optimal depth sorting network problem is for $n = 17$. We can easily compute $R(S_{17})$ (in 2 seconds) as well as its restriction to networks modulo reflection. This later set consists of only 609 networks and is a key ingredient to solving this problem, effectively reducing the search space more than 300,000-fold.

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