The essence of proofs when fibring sequent calculi

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Combining logics in an important topic in applied logics [7, 1] that raises interesting theoretical problems related to transference results. The objective is to produce a new logic from two (or more) given logics by using a meta operator – the combination mechanism. Of special interest is to investigate whether the mechanism preserves logical properties of the original logics. In general, sufficient conditions can be given for preservation.

Fibring, proposed by Gabbay in [5], is one of the most challenging mechanisms for combining logics, which includes fusion of modal logics [10] as a particular case. Fibring can be and has been investigated from a deductive point of view (mainly using Hilbert calculi [11], labelled deductive systems [8] and tableau systems [2]) and also from a model-theoretic perspective (using either an algebraic approach or a modal-like semantics [6]). Several transference results have been obtained for these constructions, namely for soundness and completeness [11], several guises of interpolation and semi-decidability.

Up to now, work on fibring sequent calculi has not been considered. A possibility (following the approach for fibring Hilbert calculi) would be to say that the fibring of two sequent calculi includes the rules of both calculi written in a schematic way. However, this definition does not put into evidence how the proofs in the fibring are related to the proofs in the given sequent calculi. Herein, we present a novel notion of fibring sequent calculi where derivation is the primitive concept and where a translation technique is used to allow the mapping of a formula of the fibring into a formula of each component. This approach is inspired upon the work on fibring of abstract proof systems [3]. In this context, preservation of cut elimination and decidability can be proved. Moreover, derivation-wise both ways of defining fibring of sequent calculi are equivalent.

Definition 1. A *signature* C is a family of sets indexed by the natural numbers. The elements of each C_k are called *constructors* or *connectives* of arity k. We say that $C \subseteq C'$ if $C_k \subseteq C'_k$ for every $k \in \mathbb{N}$.

Definition 2. Let C be a signature and $\Xi = \{\xi_n : n \in \mathbb{N}\}$ be a countable set of meta-variables. The *language* $L(C, \Xi)$ is the free algebra over C generated by Ξ. The elements of $L(C, \Xi)$ are called *formulas*.

The elements of Ξ are schema variables that will allow the definition of schematic derivations. A derivation can be obtained from a schematic derivation by replacing schema variables with formulas.

Definition 3. A *substitution* is a map $\sigma : \Xi \to L(C)$. Substitutions can be inductively extended to formulas and to sets of formulas: $\sigma(\gamma)$ is the formula where each $\xi \in \Xi$ is replaced by $\sigma(\xi)$ and $\sigma(\Gamma) = \{\sigma(\gamma) : \gamma \in \Gamma\}$.

We assume that Ξ is fixed and abbreviate $L(C, \Xi)$ to $L(C)$.

Definition 4. The fibring of the signatures C' and C'' is the family $C' \cup C''$ where $(C' \cup C'')_k = C'_k \cup C''_k$ for each k.

That is, formulas in the fibring can have a mixture of the connectives of each component logic.

Definition 5. A *sequent* over a signature C is a pair $\langle \Delta_1, \Delta_2 \rangle$, denoted by $\Delta_1 \longrightarrow \Delta_2$, where Δ_1 (the *antecedent*) and Δ_2 (the *consequent*) are multi-sets of formulas in $L(C)$.

We denote by Seq_C the set of sequents over C.

Definition 6. A *sequent calculus* is a pair $\mathcal{D} = \langle C, P \rangle$ where C is a signature and $P = \{P_\Delta : \Delta \in \wp_{\text{fin}}\mathsf{Seq}_C\}$ is a family of predicates $P_\Delta \subseteq \mathsf{Seq}_C^* \times \mathsf{Seq}_C$ satisfying the following conditions.

- Conclusion: if $P_\Delta(\omega, s)$ holds, then s is the first element in ω.
- Monotonicity: if $\Delta_1 \subseteq \Delta_2$, then $P_{\Delta_1} \subseteq P_{\Delta_2}$.
- Closure under substitution: if $P_\Delta(\omega, s)$ holds and σ is a substitution, then $P_{\sigma(\Delta)}(\sigma(\omega), \sigma(s))$ also holds.

If $P_\Delta(\omega, s)$ holds for some ω, we say that s is *derivable* from Δ in sequent calculus $\mathcal{D}$, denoted $\Delta \vdash_\mathcal{D} s$.

Example 1. The traditional presentation of a sequent calculus is via rules. A *rule* is a pair $\langle \{\theta_1, \ldots, \theta_n\}, \gamma \rangle$, indicated by

$$\frac{\theta_1 \quad \ldots \quad \theta_n}{\gamma},$$

where $\theta_1, \ldots, \theta_n$ (the premises) and γ (the conclusion) are sequents. A *presentation* of a sequent calculus is a set R of rules including structural rules (e.g. cut and weakening) and specific rules for the connectives (like $\mathsf{R} \to$).

A *derivation* of a sequent s from a set of sequents Δ is a finite sequence $\Delta_{1,1} \longrightarrow \Delta_{2,1} \ldots \Delta_{1,n} \longrightarrow \Delta_{2,n}$ of sequents such that:

- $\Delta_{1,1} \longrightarrow \Delta_{2,1}$ is s;
- for each $i = 1, \ldots, n$, one of the following holds:
  - $\Delta_{1,i} \longrightarrow \Delta_{2,i}$ is an axiom (justified by Ax), that is, $\Delta_{1,i} \cap \Delta_{2,i} \neq \emptyset$;
  - $\Delta_{1,i} \longrightarrow \Delta_{2,i} \in \Delta$ (justified by Hyp);
  - there exist a rule $r = \langle \{\theta_1, \ldots, \theta_k\}, \gamma \rangle$ and a substitution σ such that $\Delta_{1,i} \longrightarrow \Delta_{2,i} = \sigma(\gamma)$ and, for each $j = 1, \ldots, k$, there is $i < i_j \leq n$ with $\sigma(\theta_j) = \Delta_{1,i_j} \longrightarrow \Delta_{2,i_j}$ (justified by $r, i_1, \ldots, i_k$).

Let R be a set of rules and define $\mathcal{D}(R) = \langle C, P \rangle$ where $P_\Delta(\omega, s)$ holds iff ω is a derivation of s from Δ. Then $\mathcal{D}(R)$ is a sequent calculus.

Example 2. The sequent calculus $S4$ for minimal logic with an $S4$ modality (characterized by Kripke structures with a transitive accessibility relation) has two unary connectives $\Box$ and $\Diamond$ and a binary connective $\to$. It is presented by the set R_{S4} containing the usual structural rules (cut, weakening and contraction) together with the following rules for the connectives.

$$\frac{\Gamma \longrightarrow \Delta, \xi_1 \quad \xi_2, \Gamma \longrightarrow \Delta}{(\xi_1 \to \xi_2), \Gamma \longrightarrow \Delta}\ \mathsf{L}\to \qquad \frac{\xi_1, \Gamma \longrightarrow \Delta, \xi_2}{\Gamma \longrightarrow \Delta, (\xi_1 \to \xi_2)}\ \mathsf{R}\to$$

$$\frac{\Gamma, \xi_1, (\Box\xi_1) \longrightarrow \Delta}{\Gamma, (\Box\xi_1) \longrightarrow \Delta}\ \mathsf{L}\Box \qquad \frac{\Box\Gamma_1 \longrightarrow \xi_1, \Delta_1}{\Gamma_2, \Box(\Gamma_1) \longrightarrow (\Box\xi_1), \Diamond(\Delta_1), \Delta_2}\ \mathsf{R}\Box$$

$$\frac{\xi_1, \Gamma_1 \longrightarrow \Diamond(\Delta_1)}{(\Diamond\xi_1), \Box(\Gamma_1), \Gamma_2 \longrightarrow \Delta_2, \Diamond(\Delta_1)}\ \mathsf{L}\Diamond \qquad \frac{\Gamma \longrightarrow \Delta, \xi_1, (\Diamond\xi_1)}{\Gamma \longrightarrow \Delta, (\Diamond\xi_1)}\ \mathsf{R}\Diamond$$

In these rules, $\Box(\Gamma) = \{(\Box\varphi) : \varphi \in \Gamma\}$ and $\Diamond(\Gamma) = \{(\Diamond\varphi) : \varphi \in \Gamma\}$.

The following derivation ω_N shows that $\{\longrightarrow \xi_1\} \vdash_{S4} \longrightarrow (\Box\xi_1)$.

1. $\longrightarrow (\Box\xi_1)$	$\mathsf{R}\Box, 2$
2. $\longrightarrow \xi_1$	Hyp

It is worth stressing that $\not\vdash_{S4} \xi_1 \longrightarrow (\Box\xi_1)$, so allowing hypotheses in the derivations is an essential feature of our definition – as is quite well-known by people working in modal logic.

Example 3. The sequent calculus D for propositional logic with connectives $\neg$ and $\to$ together with a D modality (characterized by Kripke structures where every world can access at another one) is presented by the set R_D containing the same structural rules, the two rules $\mathsf{L}\to$ and $\mathsf{R}\to$ from the previous example, and the following four rules.

$$\frac{\Gamma \longrightarrow \Delta, \xi_1}{\Gamma, (\neg\xi_1) \longrightarrow \Delta}\ \mathsf{L}\neg \qquad \frac{\Gamma, \xi_1 \longrightarrow \Delta}{\Gamma \longrightarrow (\neg\xi_1), \Delta}\ \mathsf{R}\neg$$

$$\frac{\Gamma \longrightarrow \xi_1}{\Box(\Gamma) \longrightarrow (\Box\xi_1)}\ \mathsf{R}\Box \qquad \frac{\Gamma \longrightarrow \xi_1}{\Box(\Gamma) \longrightarrow (\Diamond\xi_1)}\ \mathsf{R}\Diamond$$

The following derivation ω_D shows that $\longrightarrow \xi_2 \vdash_D \longrightarrow (\Diamond(\xi_1 \to \xi_2))$.

1. $\longrightarrow (\Diamond(\xi_1 \to \xi_2))$	$\mathsf{Cut}, 2, 5$
2. $(\Box\xi_2) \longrightarrow (\Diamond(\xi_1 \to \xi_2))$	$\mathsf{R}\Diamond, 3$
3. $\xi_2 \longrightarrow (\xi_1 \to \xi_2)$	$\mathsf{R}\to, 4$
4. $\xi_2, \xi_1 \longrightarrow \xi_2$	Ax
5. $\longrightarrow (\Diamond(\xi_1 \to \xi_2)), (\Box\xi_2)$	$\mathsf{RW}, 6$
6. $\longrightarrow (\Box\xi_2)$	$\mathsf{R}\Box, 7$
7. $\longrightarrow \xi_2$	Hyp

Before we can define fibring of sequent calculi, we need to be able to represent "mixed" formulas (built from connectives in C' and in C'') in either component. This is done by a general mechanism of translation that takes advantage of the fact that the set of variables is infinite.

Definition 7. Let C and C' be signatures with $C \subseteq C'$ and $g : L(C') \to \mathbb{N}$ be a bijection. The *translation* $\tau_g : L(C') \to L(C)$ is defined inductively as follows:

- $\tau_g(\xi_i) = \xi_{2i+1}$ for $\xi_i \in \Xi$;
- $\tau_g(c) = c$ for $c \in C_0$;
- $\tau_g(c(\gamma'_1, \ldots, \gamma'_k)) = c(\tau_g(\gamma'_1), \ldots, \tau_g(\gamma'_k))$ for $c \in C_k$ and $\gamma'_1, \ldots, \gamma'_k \in L(C')$;
- $\tau_g(c'(\gamma'_1, \ldots, \gamma'_k)) = \xi_{2g(c'(\gamma'_1,\ldots,\gamma'_k))}$ for $c' \in C'_k \setminus C_k$ and $\gamma'_1, \ldots, \gamma'_k \in L(C')$.

The translation of a set of a formulas, a sequent or a sequent of sequents is defined in the natural way.

Definition 8. With C, C' and g as above, $\tau_g^{-1} : \Xi \to L(C')$ is the substitution such that $\tau_g^{-1}(\xi_{2i}) = g^{-1}(i)$ and $\tau_g^{-1}(\xi_{2i+1}) = \xi_i$.

From this point on, we assume g is fixed and write simply τ and τ^{-1}. It is easy to see that $\tau^{-1} \circ \tau = \mathrm{id}$ and $\tau \circ \tau^{-1} = \mathrm{id}$.

Definition 9. Let $\mathcal{D}' = \langle C', P' \rangle$ and $\mathcal{D}'' = \langle C'', P'' \rangle$ be sequent calculi given by derivations. The fibring $\mathcal{D}' \uplus \mathcal{D}''$ is the sequent calculus $\langle C' \cup C'', P \rangle$, where P_Δ is inductively defined as follows.

- if $P'_{\tau'(\Delta)}(\tau'(\omega), \tau'(s))$ holds, then $P_\Delta(\omega, s)$ also holds;
- if $P''_{\tau''(\Delta)}(\tau''(\omega), \tau''(s))$ holds, then $P_\Delta(\omega, s)$ also holds;
- for finite $\Sigma = \{s_1, \ldots, s_k\} \subseteq \mathsf{Seq}_C$, if $P_\Delta(\omega_i, s_i)$ holds for $i = 1, \ldots, k$ and $P_\Sigma(\omega_s, s)$ holds, then $P_\Delta(\omega_s \cdot \omega_1 \cdot \ldots \cdot \omega_k, s)$ holds.

In this definition, τ' and τ'' denote the translations of $L(C)$ to $L(C')$ and $L(C'')$.

The intuition is as follows: a derivation in the fibring is either a derivation in one of the components (modulo translation) or recursively built from derivations using these derivations as justifications for the hypotheses used. In particular, if $\mathcal{D}'$ and $\mathcal{D}''$ are presented by rules, then each justification Hyp occurring in ω_s is interpreted as "postponing" the proof of s_i until the point where ω_i begins.

Example 4. Consider the systems $S4$ and D defined above, with the modalities renamed $\Box'$ and $\Box''$, respectively. We can prove that

$$\vdash_{\mathcal{D}(S4) \uplus \mathcal{D}(D)} \longrightarrow (\Diamond''(\xi_2 \to (\Diamond'(\xi_1 \to (\Box'\xi_1)))))$$

considering the derivation $\sigma(\omega_D) \cdot \omega_N$, where $\sigma(\xi_1) = \xi_1$ and $\sigma(\xi_2) = (\Box'(\xi_1))$.

1. $\longrightarrow (\Diamond''(\xi_1 \to (\Box'(\xi_1))))$	$\mathsf{Cut}, 2, 5$
2. $(\Box''(\Box'(\xi_1))) \longrightarrow (\Diamond''(\xi_1 \to (\Box'(\xi_1))))$	$\mathsf{R}\Diamond'', 3$
3. $(\Box'(\xi_1)) \longrightarrow (\xi_1 \to (\Box'(\xi_1)))$	$\mathsf{R} \to, 4$
4. $(\Box'(\xi_1)), \xi_1 \longrightarrow (\Box'(\xi_1))$	Ax
5. $\longrightarrow (\Diamond''(\xi_1 \to (\Box'(\xi_1)))), (\Box''(\Box'(\xi_1)))$	$\mathsf{RW}, 6$
6. $\longrightarrow (\Box''(\Box'(\xi_1)))$	$\mathsf{R}\Box'', 7$
7. $\longrightarrow (\Box'(\xi_1))$	Hyp

1. $\longrightarrow (\Box'\xi_1)$	$\mathsf{R}\Box', 2$
2. $\longrightarrow \xi_1$	Hyp

The boxes are shown for clarity. The reader can verify that this derivation does indeed satisfy the definition of derivation in the fibring.

This example shows that this sequent calculus is equivalent to the one presented by $R_{S4} \cup R_D$. This is a general fact: the fibring of two calculi presented by rules is equivalent to the calculi presented by the union of the rules. However, the definition given captures the essence of a proof in the fibring in a much clearer way: a proof in the fibring consists of proofs in the components joined together at a higher level by concatenation. This is not the case in the system presented by the union of the rules, where derivations do not bear any relationship to the ones in the original calculi.

We conclude by stating some properties of fibring. The first result concerns the ability of removing the cut rule in a system presented by rules.

Definition 10. A sequent calculus presented by a set of rules R has cut elimination iff, for any $\Delta \subseteq \mathsf{Seq}_C$ and $s \in \mathsf{Seq}_C$, whenever $\Delta \vdash_{\mathcal{D}(R)} s$ there is a derivation ω for $\Delta \vdash_{\mathcal{D}(R)} s$ that does not use the cut rule.

Proposition 1. Let $\mathcal{D}'$ and $\mathcal{D}''$ be sequent calculi presented by the sets of rules R' and R'', respectively, with cut elimination. Then the system presented by $R = R' \cup R''$ also has cut elimination.

The proof proceeds by considering the fibring $\mathcal{D}' \uplus \mathcal{D}''$, which is equivalent to the calculus presented by R, but where derivations can only be joined by concatenation.

Another useful property of a sequent calculus is the ability to decide whether a given derivation proves a sequent from a set of hypotheses. In order for this to hold, it is reasonable to assume that the set of hypotheses is recursive.

Definition 11. A sequent calculus given by derivations $\mathcal{D} = \langle C, P \rangle$ is *decidable* iff, for every recursive set $\Delta \subseteq \mathsf{Seq}_C$, the relation P_Δ is recursive.

Proposition 2. Let $\mathcal{D}'$ and $\mathcal{D}''$ be decidable sequent calculi given by derivations. Then their fibring $\mathcal{D} = \mathcal{D}' \uplus \mathcal{D}''$ is decidable.

As a conclusion, we were able to capture the relationship between derivations in the fibring with the derivations in the component logics by introducing a novel notion of fibring sequent calculi. This notion relies upon a translation technique that allows us to map a formula of the fibring into a formula of each component.

This new notion of fibring sequent calculi is compared with a more usual one in which the sequent calculi are presented by rules. It is then shown to preserve cut elimination and decidability.

Natural extensions of this work are to consider fibring sequent calculi for display logics [4, 9] and fibring of labelled sequent calculi. Also of interest would be to extend the work to the context of logics with quantifiers.

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