# Computational Completeness of Combinatory Algebras

Luís Cruz-Filipe

June 1st, 2001

1. Introduction

2. Basic Definitions

3. Reduction

4. Representation

Definition: A *combinatory algebra* is a quadruple $<\mathcal{D},\cdot,\mathbf{K},\mathbf{S}>$ satisfying:

- $\mathcal{D}$ is a set;
- $\cdot : \mathcal{D}\times\mathcal{D} \rightarrow \mathcal{D}$ is a binary operation;
- **K**,**S** are distinct elements of $\mathcal{D}$ such that
  - **K**xy=x
  - **S**xyz=xz(yz)

The elements of $\mathcal{D}$ are called combinators.

Definition: Let $\{x_1,\ldots,x_n\}$ be a finite set of variables. The set of *terms* over $\{x_1,\ldots,x_n\}$ is inductively defined as follows:

- $x_i$ is a term over $\{x_1,\ldots,x_n\}$, for every $1 \leq i \leq n$;
- A is a term over $\{x_1,\ldots,x_n\}$, for every $A \in \mathcal{D}$;
- if $t_1$ and $t_2$ are terms over $\{x_1,\ldots,x_n\}$, then so is $t_1 t_2$.

Generic terms over $\{x_1,\ldots,x_n\}$ are denoted as $t(x_1,\ldots,x_n)$.

Definition: The *contraction* in a combinatory algebra <$\mathcal{D}$,·,**K**,**S**>,

is the binary relation $\models$ inductively defined by:

(*Axiom*) $T \models T$, for $T \in \mathcal{D}$;

(*K*) $\mathbf{K}t_1t_2 \models t_1$, for all terms $t_1$ and $t_2$;

(*S*) $\mathbf{S}t_1t_2t_3 \models t_1t_3(t_2t_3)$, for all terms $t_1$,$t_2$ and $t_3$;

(*Congruence*) If $t_1 \models t'_1$ and $t_2 \models t'_2$, then $t_1t_2 \models t'_1t'_2$.

The reflexive and transitive closure of $\models$ is called *weak reduction*

and denoted by $\rightarrow$.

Theorem: Let $<\mathcal{D},\cdot,\mathbf{K},\mathbf{S}>$ be a combinatory algebra. For every term $t(x_1,\ldots,x_n)$ over $\{x_1,\ldots,x_n\}$ there is an element $T\in\mathcal{D}$ such that, for all $A_1,\ldots,A_n\in\mathcal{D}$,

$$TA_1\ldots A_n = t(A_1,\ldots,A_n),$$

where $t(A_1,\ldots,A_n)$ is the result of uniformly substituting each $x_i$ by the constant $A_i$ in $t(x_1,\ldots,x_n)$.

The element T is said to *represent* the term t.

Definition: Let $\langle \mathcal{D}, \cdot, \mathbf{K}, \mathbf{S} \rangle$ be a combinatory algebra. A *combinatory equation* in $\mathcal{D}$ is an expression of the form

$$Tx_2 \ldots x_n = t(T, x_2, \ldots, x_n),$$

where $t(x_1, \ldots, x_n)$ is a term over $\{x_1, \ldots, x_n\}$.

Proposition: Every combinatory equation in a combinatory algebra has a solution; that is, if $Tx_2 \ldots x_n = t(T, x\_2, \ldots, x\_n)$ is a combinatory equation, then there is an element $\mathbf{T} \in \mathcal{D}$ such that, for every $A_2, \ldots, A_n \in \mathcal{D}$,

$$\mathbf{T}A_2 \ldots A_n = t(\mathbf{T}, A_2, \ldots, A_n).$$

Furthermore,

$$\mathbf{T}A_2 \ldots A_n \Rightarrow t(\mathbf{T}, A_2, \ldots, A_n).$$

Definition: An element T is said to be *solvable* iff there are $k \in \mathbb{N}$ and elements $N_1,\ldots,N_k \in \mathcal{D}$ such that $TN_1\ldots N_k = \mathbf{I}$.

Definition: Let $k>0$ and $f\colon \mathbb{N}^k \to \mathbb{N}$ be a partial function. The combinator F is said to *represent* $f$ iff the following two conditions are satisfied:

- if $f(n_1,\ldots,n_k)\downarrow$, then $F\ulcorner n_1 \urcorner \ldots \ulcorner n_k \urcorner \to \ulcorner f(n_1,\ldots,n_k) \urcorner$;
- if $f(n_1,\ldots,n_k)\uparrow$, then $F\ulcorner n_1 \urcorner \ldots \ulcorner n_k \urcorner$ is not solvable.

# References:


- Barendregt, H. P., *The Lambda Calculus*, Elsevier Science Publishers B. V., 1984
- Engeler, E., *Foundations of Mathematics*, Springer-Verlag, 1993