The Fundamental Theorem of Calculus in `Coq`

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Overview

1. Introduction

2. Extending the FTA library

3. Partial functions

4. Consequences of constructivism

5. Domain-specific tactics

6. Conclusions

The Algebraic Hierarchy

	$\mathbb{C}$	
$\lvert\cdot\rvert$	$\mathbb{R}$ (FTC)	series
	OrdFields	$\leq$
/	Fields	
*	Rings	
−	Groups	$\sum$
+	SemiGroups	
PF id,const,comp	Setoids	

Partial Functions

Let f be a partial function from A to B and x be an element of A.

In order to apply f to x we need to know that x is in the domain of f.

$$P(x) \stackrel{\text{def}}{\Leftrightarrow} x \in \mathrm{dom}(f)$$

We can identify f with a total function from $\{x \in A : P(x)\}$ to B.

```
Record PartFunct : Type :=
  {pfpred : S->Set;
   pfprwd : (pred_well_def S predG);
   pfpfun : (Build_SubCSetoid S P)->S}
```

given

```
  f : PartFunct
  x : S
  H : (pfpred f x)
```

$f(x)$ is represented by `(pfpfun f (Build_subcsetoid_crr S P x H))`

Advantages:

- Intuitive definition
- Strongly extensional, well defined

Shortcomings:

- Unnatural mixing up between setoid elements and proofs
- Expensive simplification procedure

```
Record PartFunct : Type :=
  {pfpred : S->Set;
   pfprwd : (pred_well_def S predG);
   pfpfun : (x:S)(predG x)->S;
   pfstrx : (x,y:S)(Hx:(predG x))(Hy:(predG y))
              (((partG x Hx)[#](partG y Hy))->(x[#]y))}.
```

given

```
  f : PartFunct
  x : S
  H : (pfpred f x)
```

$f(x)$ is represented by `(pfpfun f x H)`

Advantages:

- More efficient
- Proofs are kept separate from setoid elements
- More suited to automation

Shortcomings:

- Duplicates the notion of setoid function

Constructive approach to analysis

- No pointwise concepts
- Some differences in statements of definitions/theorems
- Many differences in proofs

Derivative

Classically:

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

or, equivalently,

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_y \; |x - y| < \delta \Rightarrow \left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \varepsilon$$

Constructively:

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_y \; |x - y| \leq \delta \Rightarrow |f(y) - f(x) - f'(x)(y - x)| \leq \varepsilon |y - x|$$

Rolle's Theorem:

Classically:

$$f(a) = f(b) \Rightarrow \exists_{x \in [\mathsf{min}(a,b), \mathsf{max}(a,b)]}\ f'(x) = 0$$

Constructively:

$$f(a) = f(b) \Rightarrow \forall_{\varepsilon > 0} \exists_{x \in [\mathsf{min}(a,b), \mathsf{max}(a,b)]}\ |f'(x)| \leq \varepsilon$$

Taylor's Theorem

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R(x)$$

Classically:

$$R(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Constructively:

$$R(x) \approx \frac{f^{(n+1)}(c)}{n!}(x - c)^n(x - x_0)$$

Tactics

Typical goals:

- $X \subseteq Y$, where typically Y is the domain of some function
  Auto with Hints

- f is continuous
  Auto with Hints

- $f' = g$
  Auto with Hints is not enough

$$f(x) = 3x + 4, \quad g(x) = 3$$

Reflection

- Inductive type `symbPF`

- Interpretation function $[\![\cdot]\!]$:$\mathtt{symbPF} \rightarrow [\mathbb{R} \rightarrow \mathbb{R}]$

- Symbolic derivation $'$:$\mathtt{symbPF} \rightarrow \mathtt{symbPF}$

- Lemma: for all symbolic f, $[\![f']\!] = [\![f]\!]'$

Tactic

Given f and g:

- Build s such that $[\![s]\!] = f$ Easy
- (Try to) prove that $[\![s]\!] = f$ Usually easy
- (Try to) prove that $[\![s']\!] = g$ Not trivial!
- Apply the lemma