Reasoning about Probabilistic Sequential Programs

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Motivation

- reasoning about non-deterministic programs
- new approach: truth values for formulas

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Why EPPL

- two-layered design (exogenous approach)
- classical propositional logic at the lower level
- probabilistic logic built at the higher level

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Real-closed fields

Definition

A *real closed field* is an ordered field $\mathcal{K}$ where:

- every non-negative element of the K has a square root in K;
- every polynomial of odd degree with coefficients in K has at least one solution in K.

Example

- the set of real numbers with the usual multiplication, addition and order relation;
- the set of computable real numbers with the same operations.

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Setting

- finite range D of real numbers
- finite set $\mathbf{m} = \{0, \ldots, m-1\}$ of indices
- registers $\mathbf{xM} = \{\mathbf{xm}_k \mid k \in \mathbf{m}\}$ containing real values
- registers $\mathbf{bM} = \{\mathbf{bm}_k \mid k \in \mathbf{m}\}$ containing booleans
- variables $B = \{B_k : k \in \mathbb{N}\}$ ranging over truth values
- variables $X = \{X_k : k \in \mathbb{N}\}$ ranging over D
- real-closed field $\mathcal{K}$ with set of algebraic numbers $\mathcal{A}$
- logical variables $Y = \{y_k : k \in \mathbb{N}\}$ ranging over $\mathcal{K}$

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Language

Real terms (with $c \in D$)

$$t ::= c \mid \mathbf{xm} \mid X \mid (t + t) \mid (t\, t)$$

Classical state formulas

$$\gamma ::= \mathbf{bm} \mid B \mid (t \leq t) \mid \mathrm{ff} \mid (\gamma \Rightarrow \gamma)$$

Probability terms (with $r \in \mathcal{A}$)

$$p ::= r \mid y \mid \widetilde{r} \mid (\textstyle\int \gamma) \mid (p + p) \mid (p\, p)$$

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Useful notions

Definition

An *analytical term* is a term without occurrences of probability terms.

$$a ::= r \mid y \mid \tilde{r} \mid (a + a) \mid (aa)$$

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An *analytical formula* is a formula without occurrences of probability terms.

$$\kappa ::= (a \leq a) \mid \mathtt{fff} \mid (\kappa \supset \kappa)$$

$(\Box\gamma)$ stands for the formula $((\int\gamma) = (\int \mathtt{tt}))$
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Valuations

Definition

A *valuation* is a map that provides values to the memory variables and corresponding logical variables. The set of all valuations is denoted by $\mathcal{V}$.

The denotation $[\![t]\!]_v$ of a real term t given a valuation v is defined inductively as expected.

Satisfaction $v \Vdash_c \gamma$ of a classical state formula γ by a valuation v is also defined inductively as usual.

Definition

The *extent* of a classical state formula γ in a set V of valuations is

$$|\gamma|_V = \{v \in V \mid v \Vdash_c \gamma\}.$$

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Measure functions

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A finitely additive, discrete and bounded $\mathcal{K}$-measure μ on a set X is a map from X to $\mathcal{K}^+$ such that:

- $\mu(\emptyset) = 0$;
- if $U_1 \cap U_2 = \emptyset$, then $\mu(U_1 \cup U_2) = \mu(U_1) + \mu(U_2)$.

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Interpretation

Definition

A *generalized probabilistic state* consists of a real closed field $\mathcal{K}$ and a finitely additive, discrete and finite $\mathcal{K}$-measure over $\wp\mathcal{V}$.

Given a classical formula γ we define

$$\mu_\gamma = \lambda V.\mu(|\gamma|_V).$$

Definition

Given a real closed field $\mathcal{K}$, a *$\mathcal{K}$-assignment* is a map $\rho : Y \to \mathcal{K}$.

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Denotation of probability terms

$$\begin{aligned}
[\![r]\!]^{\rho}_{K,\mu} &= r \\
[\![y]\!]^{\rho}_{K,\mu} &= \rho(y) \\
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Satisfaction of probabilistic formulas

$$(K,\mu)\rho \Vdash (p_1 \leq p_2) \quad \text{iff} \quad [\![p_1]\!]^{\rho}_{K,\mu} \leq [\![p_2]\!]^{\rho}_{K,\mu}$$

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A classical state formula γ is said to be *valid* if it holds for all valuations $v \in \mathcal{V}$.

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$((\mathbf{x1} \leq \mathbf{x2}) \wedge (\mathbf{x1} > 0)) \Rightarrow (\mathbf{x1}^2 \leq \mathbf{x2}^2)$

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A probabilistic formula η is said to be a *probabilistic tautology* if there exists a propositional tautology β such that η is obtained from β by replacing all occurrences of $\perp$ by fff, $\rightarrow$ by $\supset$ and each propositional symbol (uniformly) by a probabilistic state formula.

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$((\int(x_1 \leq x_2)) < 1) \supset (((\int(x_1 \leq x_2)) < 1) \cap \text{tt})$

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Axioms

[CTaut] $\vdash$ $(\Box\gamma)$ for each valid state formula γ

[PTaut] $\vdash$ η for each probabilistic tautology η

[RCF] $\vdash$ $\kappa^{\vec{y}}_{\vec{p}}$ for any valid analytical formula κ

[Meas∅] $\vdash$ $((\int \mathrm{ff}) = 0)$

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The axiom system of EPPL is sound: if $\vdash \eta$, *then* $\vDash \eta$.

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The central result is to show that if η is consistent then there is a model $(\mathcal{K}, \mu)\rho$ such that $(\mathcal{K}, \mu)\rho \Vdash \eta$. The decidability follows by showing that the consistency of a formula is decidable. □

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Construction of the model

1. compute the (finite) set of valuations over the memory cells and the logical variables in the sets B and X occurring in η and let this set of valuations be V;
2. let κ_1 be the analytical formula obtained from η by effectively replacing measure terms $(\int\gamma)$ by sums $\sum_{v \Vdash_c \gamma, v \in V} y_v$ where y_v represents the probability of the valuation v;
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$$s ::= \mathrm{skip} \mid \mathbf{xm} \leftarrow t \mid \mathbf{bm} \leftarrow \gamma \mid \mathrm{toss}(\mathbf{bm}, r) \mid s; s \mid \mathrm{if}\ \gamma\ \mathrm{then}\ s\ \mathrm{else}\ s$$

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$[\![\gamma]\!]_v = \mathrm{tt}$ if $v \Vdash_c \gamma$ and $[\![\gamma]\!]_v = \mathrm{ff}$ otherwise

if m is a memory cell and e is an expression of the same type, then $\delta_e^m(v)$ assigns the value $[\![e]\!]_v$ to the cell m and coincides with v elsewhere

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The denotation of a program s is a function on generalized probabilistic states.

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$$\Psi ::= \eta \mid \{\eta\}\, s\, \{\eta\}$$

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Tossed terms

Let **bm** be a memory cell, $r \in \mathcal{A}$ be a constant and p be a probabilistic term.

The term $\mathsf{toss}(\mathbf{bm}, r; p)$ is the term obtained from p by replacing every occurrence of each measure term $(\int \gamma)$ by $\widetilde{r}(\int \gamma^{\mathbf{bm}}_{\mathrm{tt}}) + (1 - \widetilde{r})(\int \gamma^{\mathbf{bm}}_{\mathrm{ff}})$.

$$
\begin{aligned}
\mathsf{toss}(\mathbf{bm}, r; r') &= r' \\
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\mathsf{toss}(\mathbf{bm}, r; (\textstyle\int \gamma)) &= (\widetilde{r}(\textstyle\int \gamma^{\mathbf{bm}}_{\mathrm{tt}}) + (1 - \widetilde{r})(\textstyle\int \gamma^{\mathbf{bm}}_{\mathrm{ff}})) \\
\mathsf{toss}(\mathbf{bm}, r; (p + p')) &= (\mathsf{toss}(\mathbf{bm}, r; p) + \mathsf{toss}(\mathbf{bm}, r; p')) \\
\mathsf{toss}(\mathbf{bm}, r; (pp')) &= (\mathsf{toss}(\mathbf{bm}, r; p)\,\mathsf{toss}(\mathbf{bm}, r; p'))
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Tossed terms

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Tossed formulas

Let **bm** be a memory cell, $r \in \mathcal{A}$ be a constant and p be a probabilistic term.

The formula $\mathrm{toss}(\mathbf{bm}, r; \eta)$ is the formula obtained from η by replacing every occurrence of each measure term $(\int \gamma)$ by $\widetilde{r}(\int \gamma^{\mathbf{bm}}_{\mathrm{tt}}) + (1 - \widetilde{r})(\int \gamma^{\mathbf{bm}}_{\mathrm{ff}})$.

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\begin{aligned}
\mathrm{toss}(\mathbf{bm}, r; \mathrm{fff}) &= \mathrm{fff} \\
\mathrm{toss}(\mathbf{bm}, r; (p \leq p')) &= (\mathrm{toss}(\mathbf{bm}, r; p) \leq \mathrm{toss}(\mathbf{bm}, r; p')) \\
\mathrm{toss}(\mathbf{bm}, r; (\eta \supset \eta')) &= (\mathrm{toss}(\mathbf{bm}, r; \eta) \supset \mathrm{toss}(\mathbf{bm}, r; \eta'))
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Conditioned terms

Let γ be classical state formula and p be a probabilistic term.
The term (p/γ) is the term obtained from p by replacing every occurrence of each measure term $(\int \gamma')$ by $(\int(\gamma' \wedge \gamma))$.

$$
\begin{aligned}
r/\gamma &= r \\
y/\gamma &= y \\
(\textstyle\int \gamma')/\gamma &= (\textstyle\int(\gamma \wedge \gamma')) \\
(p + p')/\gamma &= (p/\gamma + p'/\gamma) \\
(pp')/\gamma &= ((p/\gamma)\,(p'/\gamma))
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Axioms

[TAUT]	$\vdash \eta$	if η is an EPPL theorem
[$\int$ FREE]	$\vdash \{\kappa\}\, s\, \{\kappa\}$	if κ is an analytical formula

[SKIP]	$\vdash \{\eta\}\ \mathsf{skip}\ \{\eta\}$
[ASGR]	$\vdash \{\eta^{\mathbf{xm}}_{t}\}\ \mathbf{xm} \leftarrow t\ \{\eta\}$
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Inference rules

[SEQ] $\{\eta_0\}\, s_1\, \{\eta_1\}, \{\eta_1\}\, s_2\, \{\eta_2\} \vdash \{\eta_0\}\, s_1; s_2\, \{\eta_2\}$

[IF] $\{\eta_1\}\, s_1\, \{y_1 = (\int \gamma_0)\}, \{\eta_2\}\, s_2\, \{y_2 = (\int \gamma_0)\}$
$\vdash \{\eta_1 \curlyvee_\gamma \eta_2\}\text{if } \gamma \text{ then } s_1 \text{ else } s_2\{y_1 + y_2 = (\int \gamma_0)\}$

[ELIMV] $\{\eta_1 \cap (y = p)\}\, s\, \{\eta_2\} \vdash \{\eta_1{}^y_p\}\, s\, \{\eta_2\}$
y does not occur in p or η_2

[CONS] $\eta_0 \supset \eta_1, \{\eta_1\}\, s\, \{\eta_2\}, \eta_2 \supset \eta_3 \vdash \{\eta_0\}\, s\, \{\eta_3\}$

[OR] $\{\eta_0\}\, s\, \{\eta_2\}, \{\eta_1\}\, s\, \{\eta_2\} \vdash \{\eta_0 \cup \eta_1\}\, s\, \{\eta_2\}$

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Substitution Lemma for classical valuations

Lemma

*For any valuation $v \in \mathcal{V}$, any classical state formula γ, any memory cell m (**xm** or **bm**) and term e of the same type,*

$$v^{m}_{[\![e]\!]_v} \Vdash_c \gamma \text{ iff } v \Vdash_c \gamma^{m}_{e}.$$

Proof.

Induction on the structure of γ. □

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Substitution Lemma for assignment

Lemma

Let $(\mathcal{K},\mu)$ be a generalized probabilistic structure and ρ be a $\mathcal{K}$-assignment. Given a memory cell m and a term e of the same type, let $\mu' = \mu \circ (\delta_e^m)^{-1}$. Then

$$[\![(\textstyle\int\gamma)]\!]^{\rho}_{(\mathcal{K},\mu')} = [\![(\textstyle\int\gamma^m_e)]\!]^{\rho}_{(\mathcal{K},\mu)}$$

for any classical state formula γ.
Furthermore, for any probabilistic term p,

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Proof.

$(\delta^m_e)^{-1}(|\gamma|\nu) = |\gamma^m_e|\nu$ and hence $\mu((\delta^m_e)^{-1}(|\gamma|\nu)) = \mu(|\gamma^m_e|\nu)$.

Therefore, by definition,

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The result is extended to probabilistic terms and formulas by induction. □

Corollary

Axioms **ASGB** *and* **ASGR** *are sound.*

Substitution Lemma for assignment

Proof.

$$(\delta_e^m)^{-1}(|\gamma|_{\mathcal{V}}) = |\gamma_e^m|_{\mathcal{V}} \text{ and hence } \mu((\delta_e^m)^{-1}(|\gamma|_{\mathcal{V}})) = \mu(|\gamma_e^m|_{\mathcal{V}}).$$

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The result is extended to probabilistic terms and formulas by induction. □

Corollary

Axioms **ASGB** *and* **ASGR** *are sound.*

Substitution Lemma for probabilistic tosses

Lemma

Let (K, μ) be a generalized probabilistic structure, ρ be a $\mathcal{K}$-assignment, $r \in \mathcal{A}$ be a constant and $\mu' = \widetilde{r}\mu \circ (\delta_{\mathrm{tt}}^{\mathbf{bm}})^{-1} + (1 - \widetilde{r})\mu \circ (\delta_{\mathrm{ff}}^{\mathbf{bm}})^{-1}$.

For any classical state formula γ,

$$[\![(\textstyle\int \gamma)]\!]_{(\mathcal{K},\mu')}^{\rho} = \widetilde{r}[\![(\textstyle\int \gamma_{\mathrm{tt}}^{\mathbf{bm}})]\!]_{(K,\mu)}^{\rho} + (1 - \widetilde{r})[\![(\textstyle\int \gamma_{\mathrm{ff}}^{\mathbf{bm}})]\!]_{(K,\mu)}^{\rho}.$$

Furthermore, for any probabilistic term p,

$$[\![p]\!]_{(\mathcal{K},\mu')}^{\rho} = [\![\mathrm{toss}(\mathbf{bm}, r; p)]\!]_{(K,\mu)}^{\rho},$$

and, for any probabilistic formula η,

$$(\mathcal{K}, \mu')\rho \Vdash \eta \text{ iff } (K, \mu)\rho \Vdash \mathrm{toss}(\mathbf{bm}, r; \eta).$$

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Proof.

Let $\mu_1 = \mu \circ (\delta_{\mathrm{tt}}^{\mathbf{bm}})^{-1}$ and $\mu_2 = \mu \circ (\delta_{\mathrm{ff}}^{\mathbf{bm}})^{-1}$. Then

$$[\![(\textstyle\int\gamma)]\!]^{\rho}_{(\mathcal{K},\mu')} = \tilde{r}[\![(\textstyle\int\gamma)]\!]^{\rho}_{(\mathcal{K},\mu_1)} + (1-\tilde{r})[\![(\textstyle\int\gamma)]\!]^{\rho}_{(\mathcal{K},\mu_2)}$$

by definition. Also

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The claim for probabilistic terms and probabilistic formulas then follows by induction. □

Corollary

Axiom TOSS *is sound.*

Substitution Lemma for probabilistic tosses

Proof.

Let $\mu_1 = \mu \circ (\delta^{\mathbf{bm}}_{\mathrm{tt}})^{-1}$ and $\mu_2 = \mu \circ (\delta^{\mathbf{bm}}_{\mathrm{ff}})^{-1}$. Then

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Corollary

Axiom **TOSS** *is sound.*

Soundness of $\int$ FREE

Lemma

For any statement s, any analytical formula κ, any generalized state $(\mathcal{K}, \mu)$ and $\mathcal{K}$ assignment ρ,

$$([\![s]\!](\mathcal{K}, \mu))\rho \Vdash \kappa \text{ iff } (\mathcal{K}, \mu)\rho \Vdash \kappa.$$

Proof.

The interpretation of analytical formulas depends only on ρ. □

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The interpretation of analytical formulas depends only on ρ. □

Soundness of **IF**

Lemma

For any generalized state $(\mathcal{K},\mu)$, $\mathcal{K}$-assignment ρ and classical state formulas γ and γ',

$$[\![(\textstyle\int\gamma')/\gamma]\!]^{\rho}_{(\mathcal{K},\mu)} = [\![(\textstyle\int\gamma')]\!]^{\rho}_{(\mathcal{K},\mu_\gamma)}.$$

Furthermore, for any probability term p,

$$[\![p/\gamma]\!]^{\rho}_{(\mathcal{K},\mu)} = [\![p]\!]^{\rho}_{(\mathcal{K},\mu_\gamma)},$$

and, for any probabilistic formula η,

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Soundness of **IF**

Proof.

By definition,

$$[\![(\textstyle\int \gamma')]\!]^{\rho}_{(\mathcal{K},\mu_\gamma)} = \mu_\gamma(|\gamma'|_\mathcal{V}) = \mu(|\gamma'|_\mathcal{V} \cap |\gamma|_\mathcal{V}) = \mu(|\gamma' \wedge \gamma|_\mathcal{V}) =$$

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The claims for probabilistic terms and formulas follow by induction. □

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Soundness of **IF**

Corollary

Given probabilistic state formulas η_1 *and* η_2*, programs* s_1 *and* s_2*, variables* $y_1 \in Y$ *and* $y_2 \in Y$ *and a classical state formula* γ*,*

$$\vDash_h \{\eta_1\}\, s_1\, \{y_1 = (\textstyle\int \gamma)\} \text{ and } \vDash_h \{\eta_2\}\, s_2\, \{y_2 = (\textstyle\int \gamma)\}$$

iff, for any classical state formula γ_0*,*

$$\vDash_h \{\eta_1 \curlyvee_{\gamma_0} \eta_2\} \text{ if } \gamma_0 \text{ then } s_1 \text{ else } s_2\, \{y_1 + y_2 = (\textstyle\int \gamma)\}.$$

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Soundness of IF

Proof.

Suppose that $(\mathcal{K},\mu)\rho \Vdash \eta_1 \curlyvee_{\gamma_0} \eta_2$. Then $(\mathcal{K},\mu)\rho \Vdash \eta_1/\gamma_0$ and $(\mathcal{K},\mu)\rho \Vdash \eta_2/(\neg\gamma_0)$. Thus, $(\mathcal{K},\mu_{\gamma_0})\rho \Vdash \eta_1$ and $(\mathcal{K},\mu_{(\neg\gamma_0)})\rho \Vdash \eta_2$. Let $(\mathcal{K},\mu_1) = [\![s_1]\!](\mathcal{K},\mu_{\gamma_0})$, $(\mathcal{K},\mu_2) = [\![s_2]\!](\mathcal{K},\mu_{(\neg\gamma_0)})$ and $\mu' = \mu_1 + \mu_2$.

Since $\Vdash_h \{\eta_1\}\, s_1\, \{y_1 = (\int\gamma)\}$ and $(\mathcal{K},\mu_{\gamma_0})\rho \Vdash \eta_1$, it follows that $(\mathcal{K},\mu_1) \Vdash_h y_1 = (\int\gamma)$. Thus, by definition $\rho(y_1) = \mu_1(|\gamma|\nu)$. Similarly, $\rho(y_2) = \mu_2(|\gamma|\nu)$.

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Let $(\mathcal{K},\mu_1) = [\![s_1]\!](\mathcal{K},\mu_{\gamma_0})$, $(\mathcal{K},\mu_2) = [\![s_2]\!](\mathcal{K},\mu_{(\neg\gamma_0)})$ and $\mu' = \mu_1 + \mu_2$.
Since $\Vdash_h \{\eta_1\}\, s_1\, \{y_1 = (\int\gamma)\}$ and $(\mathcal{K},\mu_{\gamma_0})\rho \Vdash \eta_1$, it follows that $(\mathcal{K},\mu_1) \Vdash_h y_1 = (\int\gamma)$. Thus, by definition $\rho(y_1) = \mu_1(|\gamma|_\mathcal{V})$.
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Soundness of ELIMV

Lemma

Let $k = [\![p]\!]^{\rho}_{(\mathcal{K},\mu)}$ and $\rho_1 = \rho^{y}_{k}$. Then:

- *for any probabilistic term p_0, $[\![p_0]\!]^{\rho_1}_{(\mathcal{K},\mu)} = [\![{p_0}^{y}_{p}]\!]^{\rho}_{(\mathcal{K},\mu)}$;*
- *for any probabilistic formula η, $(\mathcal{K},\mu)\rho_1 \Vdash \eta$ iff $(\mathcal{K},\mu)\rho \Vdash \eta^{y}_{p}$.*

Proof.

Let p_0 be a variable y_0.
If y_0 is y, then $[\![y]\!]^{\rho_1}_{(\mathcal{K},\mu)} = k = [\![p]\!]^{\rho}_{(\mathcal{K},\mu)} = [\![y^{y}_{p}]\!]^{\rho}_{(\mathcal{K},\mu)}$.
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Soundness of **ELIMV**

Lemma

Let $k = [\![p]\!]^{\rho}_{(\mathcal{K},\mu)}$ *and* $\rho_1 = \rho^y_k$. *Then:*

- *for any probabilistic term* p_0, $[\![p_0]\!]^{\rho_1}_{(\mathcal{K},\mu)} = [\![{p_0}^y_p]\!]^{\rho}_{(\mathcal{K},\mu)}$;
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Lemma

Given y not occurring in either p or in η,

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Assume that $\Vdash_h \{\eta_1 \cap (y = p)\}\, s\, \{\eta_2\}$ and suppose that $(\mathcal{K}, \mu)\rho \Vdash \eta_1{}^y_p$.

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Soundness of the calculus

Theorem

If $\vdash \Psi$ *then* $\vDash_h \Psi$.

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By induction on the length of the derivation of $\vdash \Psi$ using the previous lemmas. □

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Preterms

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\mathrm{pt}(\mathrm{skip}, p) &= p \\
\mathrm{pt}(\mathbf{bm} \leftarrow \gamma, p) &= p_{\gamma}^{\mathbf{bm}} \\
\mathrm{pt}(\mathbf{xm} \leftarrow t, p) &= p_{t}^{\mathbf{xm}} \\
\mathrm{pt}(\mathrm{toss}(\mathbf{bm}, r), p) &= \mathrm{toss}(\mathbf{bm}, r; p) \\
\mathrm{pt}(s_1; s_2, p) &= \mathrm{pt}(s_1, \mathrm{pt}(s_2, p))
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Properties of preterms

Lemma

$$[\![\mathrm{pt}(s,\, p)]\!]^{\rho}_{(\mathcal{K},\mu)} = [\![p]\!]^{\rho}_{[\![s]\!](\mathcal{K},\mu)}.$$

Weakest preconditions

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\mathrm{wp}(s, \mathrm{fff}) & = & \mathrm{fff} \\
\mathrm{wp}(s, (p_1 \leq p_2)) & = & (\mathrm{pt}(s,\ p_1) \leq \mathrm{pt}(s,\ p_2)) \\
\mathrm{wp}(s, (\eta_1 \supset \eta_2)) & = & (\mathrm{wp}(s, \eta_1) \supset \mathrm{wp}(s, \eta_2))
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Theorem

$$(\mathcal{K}, \mu)\rho \Vdash_h \mathrm{wp}(s, \eta) \ \textit{iff}\ ([\![s]\!](\mathcal{K}, \mu))\rho \Vdash_h \eta.$$

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Theorem

$$(\mathcal{K}, \mu)\rho \Vdash_h \mathrm{wp}(s, \eta) \textit{ iff } ([\![s]\!](\mathcal{K}, \mu))\rho \Vdash_h \eta.$$

Weakest preconditions, semantically

Corollary

$$\vDash_h \{\eta'\}\, s\, \{\eta\} \textit{ iff } \vDash (\eta' \supset \mathrm{wp}(s,\eta)).$$

Proof.

($\Rightarrow$) Suppose that $\vDash_h \{\eta'\}\, s\, \{\eta\}$ and $(\mathcal{K},\mu)\rho \Vdash \eta'$.
Then $(\llbracket s \rrbracket(\mathcal{K},\mu))\rho \Vdash \eta$, hence $(\mathcal{K},\mu)\rho \Vdash \mathrm{wp}(s,\eta)$. Therefore $\vDash (\eta' \supset \mathrm{wp}(s,\eta))$.

($\Leftarrow$) Suppose that $\vDash (\eta' \supset \mathrm{wp}(s,\eta))$ and $(\mathcal{K},\mu)\rho \Vdash \eta'$.
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Weakest preconditions, semantically

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$$\models_h \{\eta'\}\, s\, \{\eta\} \;\textit{iff}\; \models (\eta' \supset \mathrm{wp}(s, \eta)).$$

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($\Rightarrow$) Suppose that $\models_h \{\eta'\}\, s\, \{\eta\}$ and $(\mathcal{K}, \mu)\rho \Vdash \eta'$.
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($\Leftarrow$) Suppose that $\models (\eta' \supset \mathrm{wp}(s, \eta))$ and $(\mathcal{K}, \mu)\rho \Vdash \eta'$.
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Lemma

For any probabilistic term p, statement s and variable y,

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Moreover, the set of theorems of the Hoare calculus is recursive.

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Completeness. Suppose that $\models_h \{\eta'\}\, s\, \{\eta\}$. Then $\models (\eta' \supset \mathrm{wp}(s,\eta))$. By completeness of EPPL, $\vdash (\eta' \supset \mathrm{wp}(s,\eta))$. On the other hand, $\vdash \{\mathrm{wp}(s,\eta)\}\, s\, \{\eta\}$, whence $\vdash \{\eta'\}\, s\, \{\eta\}$ by **CONS**.

Decidability. By soundness and completeness, $\vdash \{\eta'\}\, s\, \{\eta\}$ iff $\models_h \{\eta'\}\, s\, \{\eta\}$. By completeness of EPPL and the properties of weakest preconditions, it follows that $\vdash \{\eta'\}\, s\, \{\eta\}$ iff $\vdash (\eta' \supset \mathrm{wp}(s,\eta))$. The decidability is now a consequence of the decidability of EPPL and the fact that $\mathrm{wp}(s,\eta)$ can be computed algorithmically. □

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