optimizing a certified proof checker for a large-scale computer-generated proof

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outline

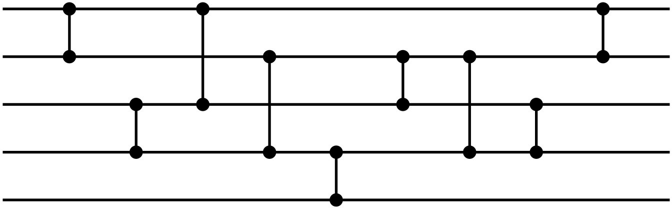
sorting networks in a nutshell

checking the result

making the checker work

conclusions & future work

a sorting network



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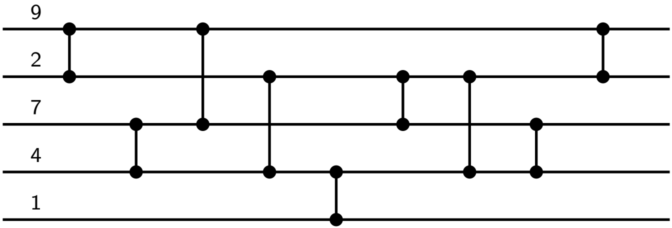

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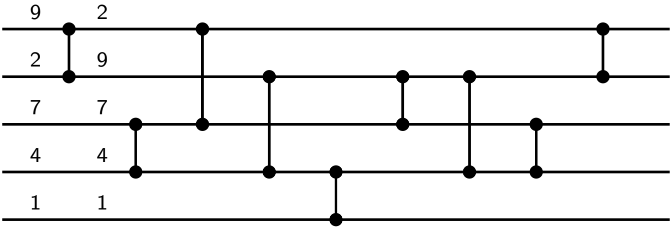

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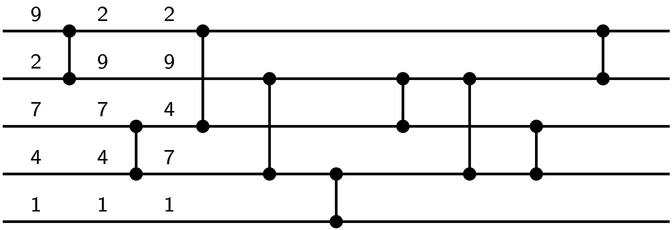

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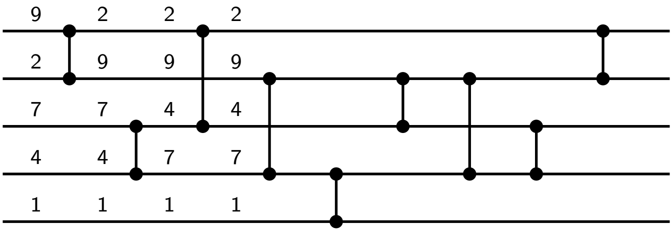

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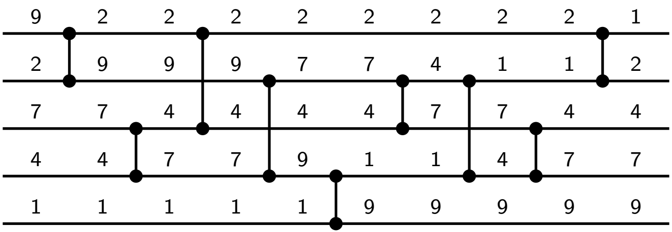

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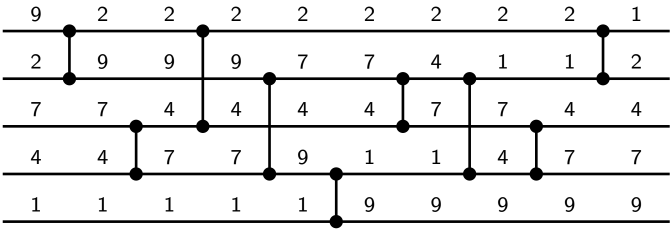

size this net has 5 *channels* and 9 *comparators*

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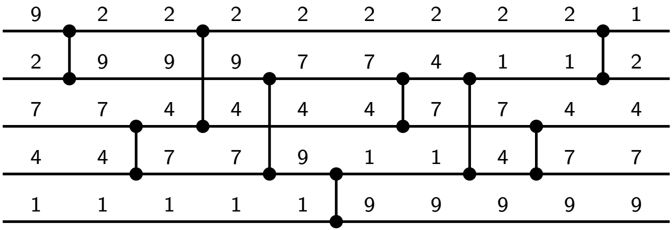

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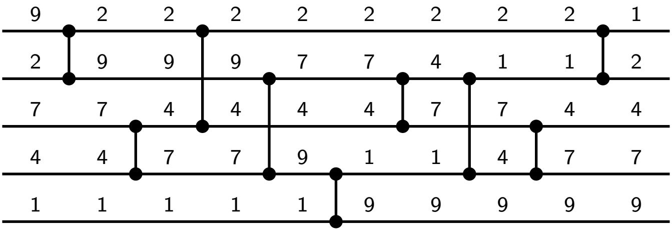

size — this net has 5 *channels* and 9 *comparators*

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the optimal size problem — what is the minimal number of *comparators* in a sorting network on n channels (s_n)?

history

optimal size

s_n: minimal number of *comparisons* to sort n inputs

knuth 1973

n	1	2	3	4	5	6	7	8	9	10
s_n	0	1	3	5	9	12	16	19	25 23	29 27

n	11	12	13	14	15	16	17
s_n	35 31	39 35	45 39	51 43	56 47	60 51	73 56

- values for $n \leq 4$ from information theory
- values for $n = 5$ and $n = 7$ by exhaustive case analysis

knuth $s_n \geq s_{n-1} + 3$ $\rightsquigarrow$ values for $n = 6, 8$

van voorhis $s_n \geq s_{n-1} + \lg(n)$ $\rightsquigarrow$ other lower bounds

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optimal size

yours truly 2014

s_n: minimal number of *comparisons* to sort n inputs

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- generate-and-prune algorithm
- intensive parallel computing
- $\sim$ 16 years of cpu time to compute s_9

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- $\sim$ 16 years of cpu time to compute s_9

- but how do we know that these results are correct?

outline

sorting networks

comparator	sequence of *comparators* (i, j) with $1 \leq i < j \leq n$
network	n is the number of channels

sorting networks

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output	$C(\vec{x})$ denotes the *output* of C on $\vec{x} = x_1 \dots x_n$
sorting network	$C(\vec{x})$ is sorted for every input $\vec{x}$
typical result	"C is a sorting network" is decidable

- program extraction $\rightsquigarrow$ haskell program (tests all inputs)
- nearly best possible algorithm (known result)
- short formalization ($\sim$ 35 lemmas)

the key result

output lemma (parberry 1991)

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definition

$C \preceq_\pi C'$ if π(outputs(C)) $\subseteq$ outputs(C')
$C \preceq C'$ if $C \preceq_\pi C'$ for some permutation π

the algorithm

init set $R_0^n = \{\emptyset\}$ and $k = 0$

repeat until $k > 1$ and $|R_k^n| = 1$

generate N_{k+1}^n extend each net in R_k^n by one comparator in all possible ways

prune to R_{k+1}^n keep only one element from each minimal equivalence class w.r.t. $\preceq^T$

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pruning

- quadratic step
- inner loop searches among all permutations typically fails
- record successful subsumptions

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certified checker using recorded subsumptions as an oracle

- replace pruning cycle by oracle calls
- skeptic approach towards oracle
- use program extraction
- verifies all cases up to s_8, requires $\sim$18 years for s_9...

checker soundness

```
Definition Oracle := list (comp_net * comp_net * (list nat)).

Inductive Answer : Set :=
  | yes : nat -> nat -> Answer
  | no : forall m n:nat, forall R:list comp_net,
         NoDup R ->
        (forall C, In C R -> length C = n) ->
        (forall C, In C R -> standard m C) -> Answer
  | maybe : Answer.

Fixpoint Generate_and_Prune (m n:nat) (O:list Oracle) :
   Answer.

Theorem GP_no : forall m n O R HR0 HR1 HR2,
   Generate_and_Prune m n O = no m n R HR0 HR1 HR2 ->
   forall C, sort_net m C -> length C > n.

Theorem GP_yes : forall m n O k,
   Generate_and_Prune m n O = yes m k ->
  (forall C, sort_net m C -> length C >= k) /\
   exists C, sort_net m C /\ length C = k.
```

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- call oracle to solve difficult tasks
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in our case

- oracle is pre-computed (offline)
- information from oracle guides algorithm
- potential for optimizations

improving the pruning step

old algorithm while oracle has a next subsumption $C \preceq_\pi C'$

1. check that $C \preceq_\pi C'$
2. check that C, C' are in the current set
3. remove C' from the current set

(laziness performs the last two steps together)

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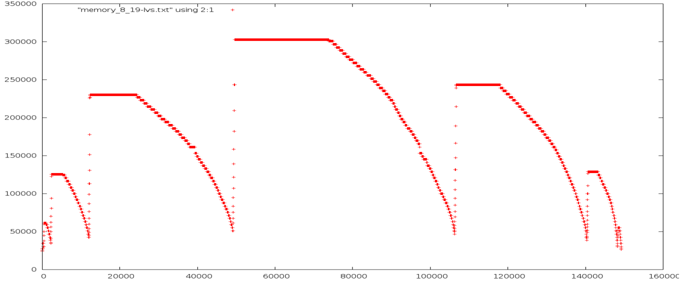

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after: check that all stored networks are in the final set

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requirement

- cannot have subsumption chains, e.g. $C_1 \preceq C_2 \preceq C_3$

pre-processing replace chains by endpoint subsumptions (e.g. $C_1 \preceq C_3$)
computing adequate permutation

$\rightsquigarrow$ don't care how, they will be checked anyway!

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optimizations

- provide C's in the order they were generated
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- store Cs in a search tree
  (improves performance)
- use search trees in some other places
  (duh?)

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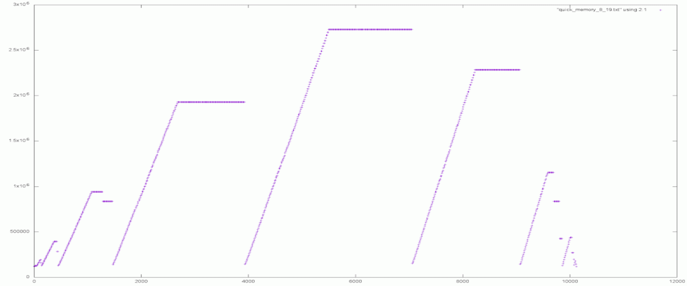

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less memory

- extract naturals to native integers
  (unfortunately necessary, but clearly sound)

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less memory

- extract naturals to native integers
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- represent comparators as a single number
  (halves memory consumption)

philosophical considerations

the good news checker verifies s_9 in around 6 days using "moderate" resources

moderate not-so-new commonplace cpu, 64 gb ram

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more good news (almost) no changes to the formalization

- relatively quick changes (a few hours each)
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offline oracles a new methodology?

outline

conclusions & future work

results

- formal verification of exact values of s_n for $n \leq 9$
- new methodology (offline oracles)
- able to deal with $\sim$ 27 gb of proof witnesses
- clean separation between formalization ("mathematics") and optimization of checker ("computer science")

next episodes

- formal proof of van voorhis' $s_n \geq s_{n-1} + \lg(n)$ to obtain s_{10}
- other problems in sorting networks
- application of this method to other search-intensive proofs

thank you!