formally proving the boolean pythagorean triples conjecture

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outline

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verifying unsatisfiability

tacas'17

certifying (unsat) results from sat solvers

- enriched trace format
- verification procedure formalized in coq
- correct-by-construction extracted checker

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evaluation

examples from the 2015 and 2016 sat competitions. . .

verifying unsatisfiability

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evaluation

examples from the 2015 and 2016 sat competitions...
...and "the large proof ever", because it's there

- unexpected success

the boolean pythagorean triples problem

a problem in ramsey theory

can the natural numbers be colored with two colors such that no pythagorean triple is monochromatic?

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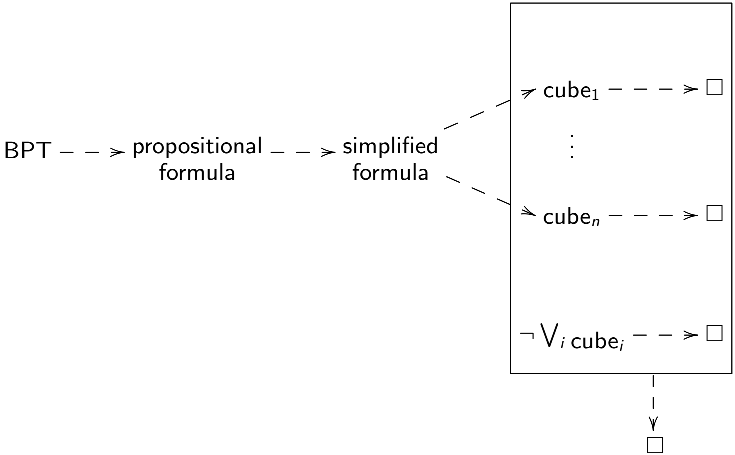
no

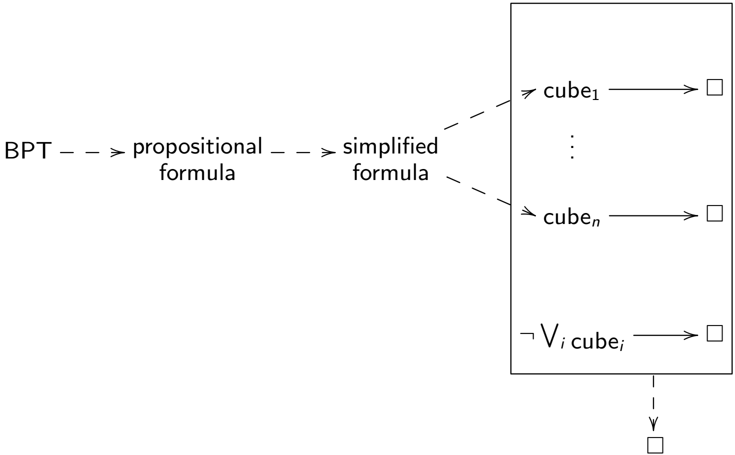
heule *et al.* showed that the finite restriction to $\{1, \ldots, 7825\}$ is already unsolvable

- encoding as a propositional formula
- simplification step
- divide-and-conquer strategy
- one million and one unsatisfiable formulas

proof strategy



proof strategy



our goal

the skeptic's view

we have shown that some 1,000,001 propositional formulas are unsatisfiable

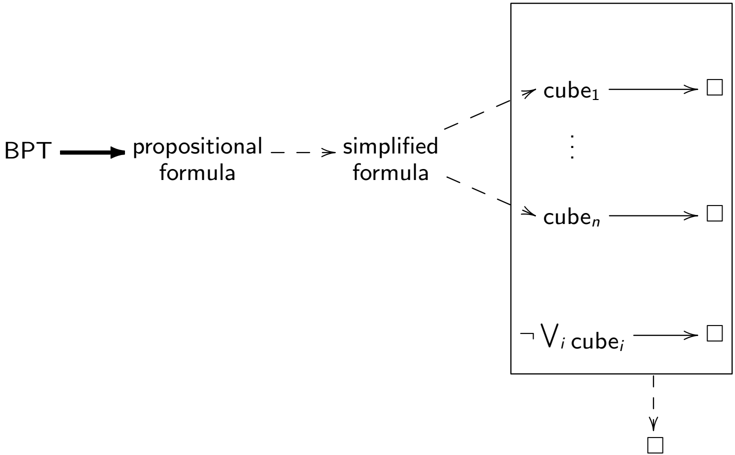
the challenge

formally verify all the steps in the process

- state the mathematical problem
- prove the propositional encoding sound
- prove the simplification steps sound
- prove the divide-and-conquer strategy sound

outline

road map


BPT
propositional formula
simplified formula
cube_1
cube_n
$\neg \bigvee_i \text{cube}_i$

the boolean pythagorean triples problem

definitions

- we use the coq type of (binary) positive numbers
- our "colors" are true and false

```
Definition coloring := positive -> bool.

Definition pythagorean (a b c:positive) := a*a + b*b = c*c.

Definition pythagorean_pos (C:coloring) := forall a b c,
  pythagorean a b c -> (C a <> C b) \/ (C a <> C c) \/ (C b <> C c).
```

a propositional encoding

```
Definition Pythagorean_formula (n:nat) := [...]
```

$$\bigwedge_{1 \leq a < b < c < n} (x_a \vee x_b \vee x_c) \wedge (\overline{x_a} \vee \overline{x_b} \vee \overline{x_c})$$

- (some) direct encoding in functional programming (we first build a list of pythagorean triples)
- n should be 7826, but it pays off to leave it uninstantiated

a propositional encoding

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- (some) direct encoding in functional programming
  (we first build a list of pythagorean triples)
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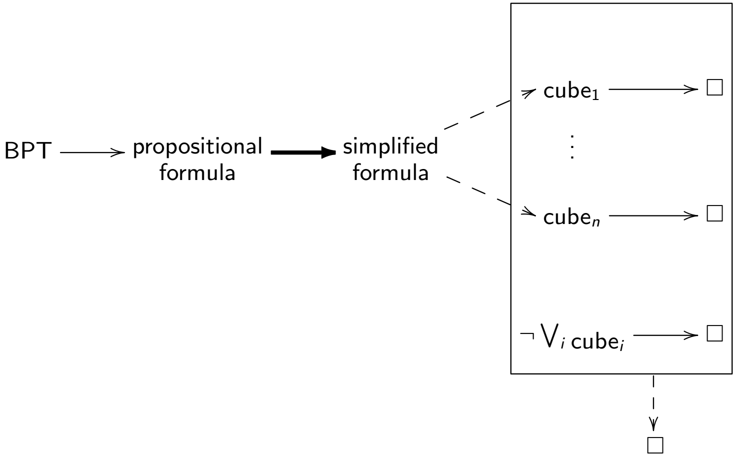
```
Parameter TheN : nat.

Definition The_CNF := Pythagorean_formula TheN.

Theorem Pythagorean_Theorem : unsat The_CNF -> forall C, ~pythagorean_pos C.
```

- we can extract to ml and recompute the propositional formula

road map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blocked clause elimination (i/ii)

in general

reduce the size of a cnf by eliminating clauses that do not change satisfiability

in this case

if k occurs in exactly one pythagorean triple, then that triple can be removed from the set

- any coloring that makes all remaining triples monochromatic can be trivially extended to k

blocked clause elimination (ii/ii)

```
Fixpoint simplify (t:triples) (l:list positive) := match l with
  | nil => t
  | p::l' => if (one_occurrence_dec p t) then simplify (remove_number p t) l'
             else simplify t l'
  end.

Definition simplified_Pythagorean_formula (n:nat) (l:list positive) := [...]

Parameter The_List : list positive.

Definition The_Simple_CNF := simplified_Pythagorean_formula TheN The_List.

Theorem simplification_ok : unsat The_CNF <-> unsat The_Simple_CNF.
```

- The_List is instantiated by a concrete list built from the trace of heule *et al.*'s proof

the symmetry break (i/ii)

idea

add additional constraints that preserve satisfiability but reduce the number of solutions
(“without loss of generality...”)

concretely

impose that 2520 is colored `true`

- nothing magical about 2520
- it just happen to be the number occurring most often

the symmetry break (ii/ii)

```
Lemma fix_one_color : forall C, pythagorean_pos C ->
  forall n b, exists C', pythagorean_pos C' /\ C' n = b.

Parameter TheBreak : positive.

Definition The_Asymmetric_CNF := [...]

Theorem symbreak_ok : unsat The_CNF <-> unsat The_Asymmetric_CNF.
```

- `The_Asymmetric_CNF` simply has the extra clause x_{2520}
- using program extraction we can compute the simplified propositional formula in approx. 35 minutes

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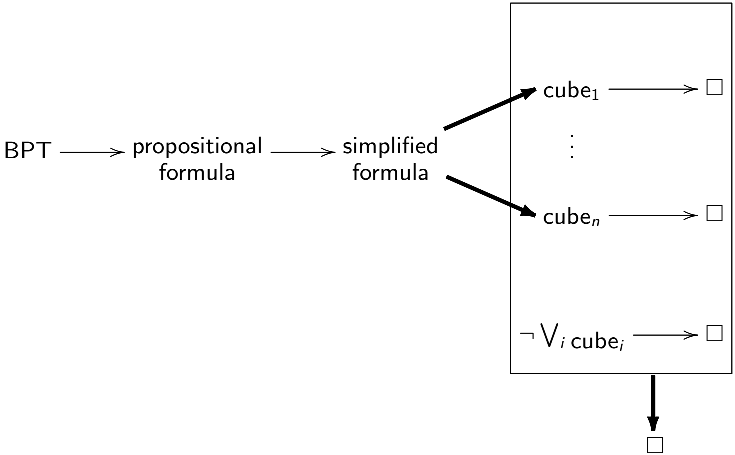
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$\neg \bigvee_i$ cube$_i$

cube-and-conquer

methodology

find a set of partial valuations (the cubes) such that:

- the conjunction of the cnf with each cube is unsatisfiable
- the disjunction of the cubes is a tautology

a perfect balance

cubes are built using heuristics

- replace one big problem with many smaller ones
- need criteria to decide when to stop splitting
- not our problem!

cube-and-conquer, coq style

```
Definition Cube := list Literal.

Fixpoint Cubed_CNF (F:CNF) (C:Cube) : CNF := [...]

Fixpoint noCube (C:list Cube) : CNF := [...]

Lemma CubeAndConquer_lemma : forall Formula Cubes,
  (forall c, In c Cubes -> unsat (Cubed_CNF Formula c)) ->
  unsat (noCube Cubes) -> unsat Formula.
```

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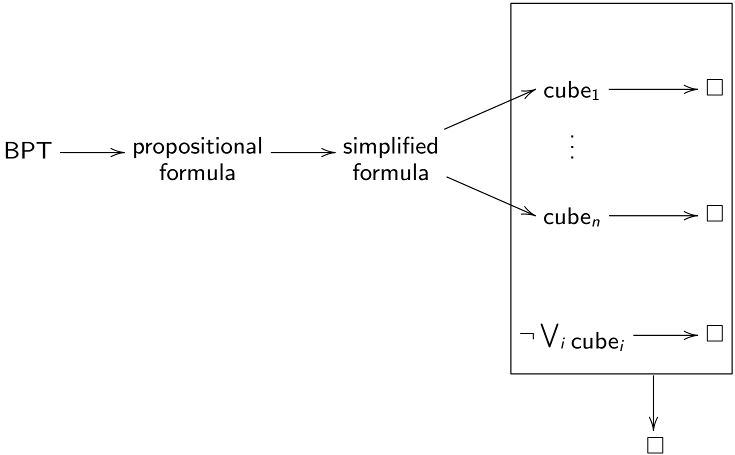
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plugging it all together

in this work

needed to reuse results from tacas'17

- no resources to rerun all unsatisfiability proofs
- additional steps to connect to previous results

afterwards

refactored the source code

- can now be run in one go (at your own risk)
- hid some nasty details

outline

conclusions

- formally verified unsolvability of the boolean pythagorean triples problem
- stronger claim for the mathematical result
- formal generation of the propositional encoding
- take-home lesson: this is not so hard. . .

thank you!