Notes on Mader's $\mathcal{S}$-Paths Theorem

ASP

December 9, 2008

1 Introduction

These (informal) notes include a proof of Mader's $\mathcal{S}$-Paths Theorem [4]. The proof presented here is based on Schrijver's proof [6][1], but tries to go through it in greater detail. In particular, I shall only write 'clearly', 'obviously', etc. when I truly believe that it is so. Section 3 includes a self-contained induction proof of the Tutte-Berge formula as well as an argument showing how to derive the Tutte-Berge formula from Mader's $\mathcal{S}$-Paths Theorem.

Let $G = (V, E)$ denote an undirected, simple and finite graph. Let $\mathcal{S}$ denote a collection of non-empty disjoint subsets of V. The sets of $\mathcal{S}$ will be referred to as $\mathcal{S}$-sets. An $\mathcal{S}$-path (of G) is a path in G with end vertices in two distinct $\mathcal{S}$-sets. Throughout the proof we let S denote the union of all $\mathcal{S}$-sets. A partition of a set M is a set $\mathcal{M}$ of disjoint subsets of M whose union is M. Note that this definition of the term partition does not exclude the empty set $\emptyset$ being one of the elements of $\mathcal{M}$. A *feasible partition* $U_0, \ldots, U_n$ (of V w.r.t. G and $\mathcal{S}$) is a partition $U_0, \ldots, U_n$ of V with the property that any $\mathcal{S}$-path disjoint from U_0 traverses some edge spanned by some U_i.

Theorem 1.1 (Mader [4]). *The maximum number of disjoint $\mathcal{S}$-paths is equal to the minimum value of*

$$|U_0| + \sum_{i=1}^{n} \left\lfloor \frac{|B_i|}{2} \right\rfloor, \tag{1}$$

taken over all feasible partitions $U_0, \ldots, U_n$ of V. Here B_i denotes the set of vertices of U_i that belong to S or have (at least) one neighbour in $V \setminus (U_0 \cup U_i)$.

We shall, in some cases during the proof, need to determine the sets B_i; note that $U_i \cap S \subseteq B_i \subseteq U_i$.

[1] A preprint of the paper [6] is freely available at Schrijver's homepage [1].

(Question: Does there exist any feasible partition of V w.r.t. G and $\mathcal{S}$? Yes, at least one. Just take $U_0 = V$.)

Let μ denote the minimum obtained in (1).

Note that if $\mathcal{S}$ contains just one element, then there is no $\mathcal{S}$-path, and in this case we obtain zero in (1) by choosing $U_0 := \emptyset$ and $U_i := \{v_i\}$, where $V = \{v_1, v_2, \ldots, v_n\}$.

For any integer $k \geq 1$, let $[k]$ to denote the set $\{1, 2, \ldots, k\}$.

2 Proof of Mader's $\mathcal{S}$-Paths Theorem

Proof. We shall first show that the maximum number of disjoint $\mathcal{S}$-sets is bounded above by μ.

Let $U_0, \ldots, U_n$ denote some feasible partition, and suppose P denotes an $\mathcal{S}$-path disjoint from U_0. Then, by definition of the concept of feasible partitions, P traverses an edge $e = xy$, where $x, y \in U_i$ for some $i \in [n]$. Let the end vertices of P be denoted a and b such that when starting from a going towards b along P we get to x before we get to y.

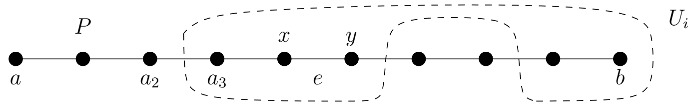

Figure 1: Illustration for the argument of Obs. 2.1.

Observation 2.1. *The path P contains at least two vertices of B_i.*

Argument. If a (resp. b) is in U_i, then a (resp. b) is in B_i, since the end vertices of P are in S. Thus, if $a, b \in U_i$, then a and b are distinct vertices of P in B_i. If $a \notin U_i$, then there is vertex $a_2 \in V \backslash (U_0 \cup U_i)$ on the $a - x$ subpath of P which is adjacent to a vertex $a_3 \in U_i$ and $a_3 \in V(P) \cap B_i$. If $b \in U_i$, then a_3 and b are two distinct vertices of P in B_i (see Figure 1). If $b \notin U_i$, then there is a vertex $b_2 \in V \backslash (U_0 \cup U_i)$ on the $b - y$ subpath of P which is adjacent to a vertex $b_3 \in U_i$, and so $b_3 \in B_i$. Again, we have two distinct vertices, a_3 and b_3, of P in B_i.

$\diamond$

Let $\mathcal{P}$ denote some set of disjoint $\mathcal{S}$-sets all disjoint from U_0. Now, according to Observation 2.1, for each path $P \in \mathcal{P}$ there exists a set B_i (as defined above) containing two vertices of P; let's call these two vertices of P the *representative pair* of P. Given some integer $i \in [n]$, how many representative pairs for paths in $\mathcal{P}$ do we find in B_i? Obviously, at most $\lfloor |B_i|/2 \rfloor$. Thus, the number of paths in $\mathcal{P}$ is bound above by

$$\sum_{i=1}^{n} \left\lfloor \frac{|B_i|}{2} \right\rfloor.$$

It follows that the number of disjoint $\mathcal{S}$-paths is at most

$$|U_0| + \sum_{i=1}^{n} \left\lfloor \frac{|B_i|}{2} \right\rfloor,$$

and, since this holds for every feasible partition of V w.r.t. G and $\mathcal{S}$, we obtain the desired upper bound (1).

The task in the remainder of the proof is to prove the existence of μ disjoint $\mathcal{S}$-paths.

By the statement just after the theorem we may assume that the set $\mathcal{S}$ contains at least two elements.

Part I. Suppose that $|T| = 1$ for all sets $T \in \mathcal{S}$. In this case, Schrijver employs a method of Gallai [3], in which the problem is translated into a problem in matching theory.

Let G' denote a copy of $G - S$. (That is, G' is isomorphic to $G - S$, but $VG') \cap V(G) = \emptyset$) Given some vertex $v \in V \backslash S$, let v' denote the copy of v in G'. Construct the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ from the union of G and G' by adding (1) the edges vv' for every vertex of $v \in V \backslash S$ and (2) edges such that $N(v, \tilde{G}) \backslash \{v'\} = N(v', \tilde{G}) \backslash \{v\}$; see Figure 2.

Define $N := \{vv' \mid v \in V \backslash S\}$. Then, obviously, N constitute a perfect matching of $\tilde{G} - S$.

Claim 2.2. *If $\tilde{G}$ contains a matching M of size $\mu + |V \backslash S|$, then we can find μ disjoint $\mathcal{S}$-paths in G.*

Given some subset $F \subseteq E$ of edges, an F-edge is simply an edge contained in F.

Proof of Claim 2.2. Suppose $\tilde{G}$ contains a matching M of size $\mu + |V \backslash S|$. Let's take a look at the induced graph $\tilde{G}[M \cup N]$. (Only vertices incident with

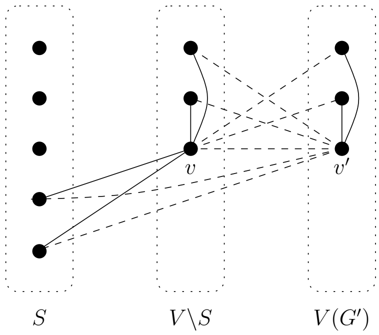

Figure 2: The graph $\tilde{G}$ defined in Part I of the proof. The edges indicated by dashed lines are new edges introduced to the union $G \cup G'$ such that in the resulting graph $\tilde{G}$ we obtain $N(v, \tilde{G})\backslash\{v'\} = N(v', \tilde{G})\backslash\{v\}$.

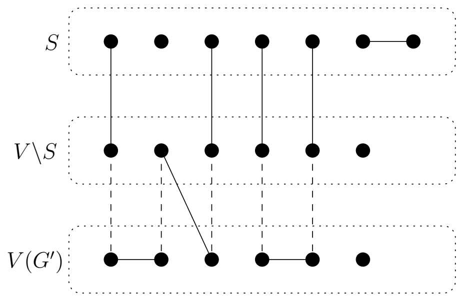

Figure 3: The graph $\tilde{G}$ with the paths of $\mathcal{C}$ indicated in the obvious way. M-edges are drawn as solid lines, while N-edges are drawn as dashed lines.

an edge of $M \cup N$ occur in $\tilde{G}[M \cup N]$.) Obviously, the vertices of $\tilde{G}[M \cup N]$ have degree one or two, which implies that the components of $\tilde{G}[M \cup N]$ are paths and cycles. There are basically three types of components of $\tilde{G}[M \cup N]$: The components containing the same number of M-edges as N-edges, the components containing one more N-edge than M-edge, and the components containing one more M-edge than N-edge. As $|M| = \mu + |V\backslash S| = \mu + |N|$, there exists a set $\mathcal{C}$ of exactly μ components of this latter type. A component with one more M-edge than N-edge in $\tilde{G}[M \cup N]$ is non-trivial path P with both end vertices, say x and y, incident with M-edges. Since P is a component in the induced graph $\tilde{G}[M \cup N]$, and N is a perfect matching between $V\backslash S$ and $V(G')$, it follows that both x and y are in S (See Figure 3).

Now contracting the edges of N in the paths of $\mathcal{C}$ yields a set of μ disjoint $\mathcal{S}$-sets in G. (Here we actually used the fact that the $\mathcal{S}$-sets all have size 1, since we need the paths to connect to *distinct* $\mathcal{S}$-sets) Thus, we got what we needed. $\square$

According to Claim 2.2, we only need to show that $\tilde{G}$ has a matching of size $\mu + |V\backslash S|$.

Claim 2.3. *The graph $\tilde{G}$ has a matching of size $\mu + |V\backslash S|$.*

A component is said to be even (resp. odd), if it has even (resp. odd) order. Given some graph H, we let $o(H)$ denote the number of odd components of H.

Proof of Claim 2.3. We shall use the Tutte-Berge formula (see Section 3), which states that the cardinality of maximum matching in $\tilde{G}$ is equal to

$$\frac{1}{2} \min_{A \subseteq V(\tilde{G})} \left(|V(\tilde{G})| + |A| - o(\tilde{G} - A) \right). \tag{2}$$

Suppose that for any $\tilde{U_0} \subseteq V(\tilde{G})$,

$$|\tilde{U_0}| + \sum_{i=1}^{n} \left\lfloor \frac{|\tilde{U_i}|}{2} \right\rfloor \geq \mu + |V\backslash S|, \tag{3}$$

where $\tilde{U_1}, \tilde{U_2}, \ldots, \tilde{U_n}$ are the components of $\tilde{G} - \tilde{U_0}$.

We get the following identity by considering the parity of the numbers $|\tilde{U_i}|$ for $i \in [n]$.

$$\sum_{i=1}^{n} \left\lfloor \frac{|\tilde{U_i}|}{2} \right\rfloor = \frac{1}{2}|V(\tilde{G}) - \tilde{U_0}| - \frac{1}{2} o(\tilde{G} - \tilde{U_0}) \tag{4}$$

Hence

$$\begin{aligned} \frac{1}{2}\left(|V(\tilde{G})| + |\tilde{U_0}| - o(\tilde{G} - \tilde{U_0}) \right) &= |\tilde{U_0}| + \sum_{i=1}^{n} \left\lfloor \frac{|\tilde{U_i}|}{2} \right\rfloor \\ &\geq \mu + |V\backslash S|, \end{aligned} \tag{5}$$

where the last inequality follows from (3). Now the inequality in (5) and the Tutte-Berge formula (2) implies that $\tilde{G}$ has a matching of size $\mu + |V\backslash S|$. Thus, we need only show that (3) holds for every subset $\tilde{U_0}$ of $V(\tilde{G})$.

If, for some vertex $v \in V\backslash S$, exactly one of v and v' belong to $\tilde{U_0}$, then remove that vertex (v or v') from $\tilde{U_0}$ thereby not increasing the left-hand side of (3), that is,

$$|\tilde{U_0}| + \sum_{i=1}^{n} \left\lfloor \frac{|\tilde{U_i}|}{2} \right\rfloor.$$

Let's see why. Obviously, $|\tilde{U_0}|$ drops by one, but what about $\sum \left\lfloor \frac{|\tilde{U_i}|}{2} \right\rfloor$?

Suppose w.l.o.g. $v \in \tilde{U_0}$ and $v' \in \tilde{U_j}$ for some $j \in [n]$. Now, since $N(v, \tilde{G})\backslash\{v'\} = N(v', \tilde{G})\backslash\{v\}$, we find that the components of $\tilde{G} - (\tilde{U_0}\backslash\{v\})$ and $\tilde{G} - \tilde{U_0}$ are the same except the component $\tilde{U_j}$ in the former graph has been replaced by the component $\tilde{G}[V(\tilde{U_j}) \cup \{v\}]$. Thus, only $|\tilde{U_j}|$ is replaced by a larger value, and, since it is replaced by $|\tilde{U_j}|+1$, it follows that $\sum \left\lfloor \frac{|\tilde{U_i}|}{2} \right\rfloor$ increases by at most one.

Hence it suffices to show that (3) holds for all subsets $\tilde{U_0}$ of $V(\tilde{G})$ with the property that for every vertex $v \in V\backslash S$, we have $v, v' \in \tilde{U_i}$ for some $i \in \{0, 1, \ldots, n\}$.

Suppose that we are given some arbitrary subset $\tilde{U_0}$ of $V(\tilde{G})$.

Fact 2.4. *We may assume that for every vertex $v \in V\backslash S$, we have $v, v' \in \tilde{U_i}$ for some $i \in \{0, 1, \ldots, n\}$.*

Let $U_i := \tilde{U_i} \cap V$ for $i \in \{0, 1, \ldots, n\}$. Recall, that $\tilde{U_1}, \ldots, \tilde{U_n}$ denotes the components of $\tilde{G} - \tilde{U_0}$, in particular, there is no edge between any vertex of $\tilde{U_i}$ and $\tilde{U_j}$ for $i, j \in [n]$ and $i \neq j$. Thus, obviously each component of $G - U_0$ is a subgraph of some component $\tilde{U_j}$ ($j \in [n]$), in particular, it is a subgraph of U_j. Is it possible that U_j is disconnected? For any pair of vertices $x, y \in U_j$, there is an (x, y)-path P in $\tilde{U_j}$. Now if P uses any vertices $w' \in \tilde{U_j}\backslash U_j$ (recall, that w' denotes the vertex of $V(G')$ which is the unique copy of the vertex w in $V\backslash S$), then, according to Fact 2.4, the corresponding vertex w of $V\backslash S$ is in U_j. Moreover, since $N(v, \tilde{G})\backslash\{v'\} = N(v', \tilde{G})\backslash\{v\}$, it follows that vertices w' of P in $\tilde{U_j}\backslash U_j$ may be replaced by the corresponding vertices w to produce an (x, y)-walk completely contained in $\tilde{U_j}$. This shows that U_j is connected, and so $U_1, \ldots, U_n$ are precisely the components of $G - U_0$.

Next, we need to do a bit of computation. Obviously, $|\tilde{U_0}| = |U_0| + |\tilde{U_0} \cap V(G')|$.

$$
\begin{aligned}
\beta &:= |\tilde{U_0}| + \sum_{i=1}^{n} \left\lfloor \frac{|\tilde{U_i}|}{2} \right\rfloor \\
&= |U_0| + |\tilde{U_0} \cap V(G')| + \sum_{i=1}^{n} \frac{1}{2} |\tilde{U_i} \cap ((V\backslash S) \cup V(G'))| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2} |U_i \cap S| \right\rfloor,
\end{aligned}
$$

where the sum $\sum \left\lfloor \frac{|\tilde{U_i}|}{2} \right\rfloor$ was split up into two sums using Fact 2.4, which implies that each set $\tilde{U_i}$ contains an even number of vertices from $(V\backslash S) \cup V(G')$.

Recall, that what we need to do is prove that β is at least $\mu + |V \backslash S|$. Let's take a closer look at one of the terms occurring above.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{2}|\tilde{U}_i \cap ((V\backslash S) \cup V(G'))\,| &= \sum_{i=1}^{n} |\tilde{U}_i \cap (V\backslash S)| \\
&= |(V\backslash S)\backslash \tilde{U}_0| \\
&= |(V\backslash S)\backslash U_0| \\
&= |V\backslash S| - |(V\backslash S) \cap U_0| \\
&= |V\backslash S| - |\tilde{U}_0 \cap V(G')|,
\end{aligned}$$

where, again, the last equality follows from Fact 2.4. Thus,

$$\begin{aligned}
\beta &= |U_0| + |\tilde{U}_0 \cap V(G')| + \left(\sum_{i=1}^{n} \frac{1}{2}|\tilde{U}_i \cap ((V\backslash S) \cup V(G'))\,|\right) + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|U_i \cap S| \right\rfloor \\
&= |U_0| + |\tilde{U}_0 \cap V(G')| + \Big(|V\backslash S| - |\tilde{U}_0 \cap V(G')|\Big) + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|U_i \cap S| \right\rfloor \\
&= |U_0| + |V\backslash S| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|U_i \cap S| \right\rfloor
\end{aligned}$$

Thus, we only need show that $|U_0| + \sum \left\lfloor \frac{1}{2}|U_i \cap S| \right\rfloor$ is at least μ.

Observation 2.5.

$$|U_0| + \sum_{i=1}^{n} \left\lfloor \frac{1}{2}|U_i \cap S| \right\rfloor \geq \mu \tag{6}$$

Argument. We just have to show that the left-hand side of (6) is one of the values over which the minimum value μ is taken.

(1) The vertex sets of the components $U_0, U_1, \ldots, U_n$ form a partition of V.

(2) Each $\mathcal{S}$-path disjoint from U_0 traverses some edge spanned by some U_i (simply because $U_1, \ldots, U_n$ are the components of $G - U_0$).

(3) By definition, $U_i \cap S \subseteq B_i \subseteq U_i$. Might a vertex $w \in U_i \backslash S$ be in B_i? No, because if that were the case, then w, by definition of B_i, would be adjacent to some vertex in $V\backslash(U_0 \cup U_i)$, which is impossible, since U_i is a component of $G - U_0$. This shows that B_i is identical to $U_i \cap S$ for each $i \in [n]$.

◇

Thus, the left-hand side of (6) is one of the values occurring in (1), and so the left-hand side is at least μ, the minimum of all such values. This, finally, establishes (3), completes the proof of Claim 2.3, and, thus, the proof of Part I. $\square$

Part II. We shall apply *reductia ad absurdum.* Fixing V, choose a counterexample $(E, \mathcal{S})$ consisting of an edge set E on V and a subpartition $\mathcal{S}$ of V such that

$$|E| - |\{\{t,u\}|\exists T, U \in \mathcal{S} : t \in T, u \in U, T \neq U\}|. \tag{7}$$

is minimized. Since the graph G is finite, the minimum in (7) exists[2]. By part I, we know that there exists at least one set $T \in \mathcal{S}$ with $|T| \geq 2$.

Claim 2.6. *The set T is an independent set of G.*

Proof of Claim 2.6. Suppose that T is not an independent set of G, choose some edge $e \in E(G[T])$, and consider $E' := E\backslash\{e\}$ and $\mathcal{S}$. Now, clearly, the value of (7) for $(E', \mathcal{S})$ is one less than the value of (7) for $(E, \mathcal{S})$. However, as we shall see, the maximum and minimum of (1) remain the same. Let G' denote the graph (V, E').

Clearly, the maximum number of $\mathcal{S}$-paths in G' is the same as the maximum number of $\mathcal{S}$-paths in G, since when considering the *maximum* number of $\mathcal{S}$-paths there is no need to use the edge e which is an 'internal' edge of the $\mathcal{S}$-set T.

Observation 2.7. *A partition $U_0, \ldots, U_n$ of V is a feasible partition w.r.t. G if and only if it is a feasible partition w.r.t. G'.*

Argument. To see this, suppose $U_0, \ldots, U_n$ is a feasible partition w.r.t. G. Any $\mathcal{S}$-path P in G' is obviously also an $\mathcal{S}$-path in G. Thus, if P is disjoint from U_0, then P contains an edge of some $G[U_i]$ ($i \neq 0$) and this edge is also in $G'[U_i]$. Conversely, suppose $U_0, \ldots, U_n$ is a feasible partition w.r.t. G', and let P denote an $\mathcal{S}$-path in G. If $e \in E(P)$, then $P - e$ contains a subpath Q which is an $\mathcal{S}$-path. If $e \notin E(P)$, let $Q := P$. Then Q (and P) contains an edge f from some set $G'[U_i]$ ($i \neq 0$), and f is also an edge of $G[U_i]$.

$\diamond$

Observation 2.8. *For every feasible partition $U_0, \ldots, U_n$ of V and for every $i \in [n]$, $B_i = B'_i$, where B_i (resp. B'_i) denotes the set of vertices of U_i which belongs to S or has a neighbour in $V\backslash(U_0 \cup U_i)$ in G (resp. G').*

[2]Here we use finiteness of G.

Argument. The only difference between G and G' is is that G' does not contain the edge e, say $e = xy$. Since the $\mathcal{S}$-sets are the same for both G and G'. the question is just what happens to the vertices x and y. Suppose w.l.o.g. $x \in U_j$. Then, since $x \in T \in \mathcal{S}$, we get $x \in B_j$ and $x \in B'_j$. Similarly for the vertex y. Hence the sets B_i and B'_i are identical for all $i \in [n]$.

◇

Now from Observations 2.7 and 2.8 it follows that the minimum in (1) (w.r.t. $\mathcal{S}$) is the same for both G and G'. Thus, $(E', \mathcal{S})$ is a counterexample with a smaller value in (7), which contradicts our choice of $(E, \mathcal{S})$. Hence T must be an independent set of G. □

Choose some vertex $s \in T$ and define

$$\mathcal{S}' := (\mathcal{S} \backslash \{T\}) \cup \{T \backslash \{s\}, \{s\}\}. \tag{8}$$

(Since $|T| \geq 2$, the set $T \backslash \{s\}$ contains at least one vertex.)

Let S' denote the union of all $\mathcal{S}'$-sets. Then, obviously, $S' = S$.

Now, clearly, the value of (7) for $(E, \mathcal{S}')$ is $|T| - 1$ less than the value of (7) for $(E, \mathcal{S})$. Hence $(E, \mathcal{S}')$ is not a counterexample to the desired theorem. Now the question is, how many disjoint $\mathcal{S}'$-paths do we have?

The minimum in (1) w.r.t. $\mathcal{S}'$ is not less than the minimum in (1) w.r.t. $\mathcal{S}$. Let's see why. Suppose $U_0, \ldots, U_n$ is a feasible partition of V w.r.t. $\mathcal{S}'$. Then it is also a feasible partition w.r.t. $\mathcal{S}$ (just think about it for a second!), and, since $S = S'$ and the edge set remain the same, we get $B'_i = B_i$.

Recall, that μ denotes the minimum obtained in (1) w.r.t. $\mathcal{S}$. Since the minimum in (1) w.r.t. $\mathcal{S}'$ is not less than μ, and $(E, \mathcal{S}')$ isn't a counterexample, it follows that there exists a collection $\mathcal{P}$ of μ disjoint $\mathcal{S}'$-paths. Clearly, we may assume that no path in $\mathcal{P}$ has any internal vertices in $S' = S$.

Fact 2.9. *No path in $\mathcal{P}$ has any internal vertices in $S' = S$.*

There must be a path $P_0 \in \mathcal{P}$ connecting s to a vertex in $T \backslash \{s\}$, since otherwise we would have μ disjoint $\mathcal{S}$-paths. This implies that all paths of $\mathcal{P} \backslash \{P_0\}$ are $\mathcal{S}$-paths.

Fact 2.10. *Every path of $\mathcal{P} \backslash \{P_0\}$ are $\mathcal{S}$-paths.*

Let u denote an internal vertex of P_0. (There must be such a vertex since T is an independent set of G) Now, define

$$\mathcal{S}'' := (\mathcal{S} \backslash \{T\}) \cup \{T \cup \{u\}\}. \tag{9}$$

Let S'' denote the union of the sets of $\mathcal{S}''$.

By the statement just after the theorem we may assume that the set $\mathcal{S}$ contains at least two sets, and so the value of (7) for $(E, \mathcal{S}'')$ is less than the value of (7) for $(E, \mathcal{S})$. Hence $(E, \mathcal{S}'')$ is not a counterexample to the desired theorem.

The minimum in (1) w.r.t. $\mathcal{S}''$ is not less than the minimum in (1) w.r.t. $\mathcal{S}$. Again, let's see why. Suppose $U_0'', U_1'', \ldots, U_n''$ denote a feasible partition w.r.t. $\mathcal{S}''$, and let P denote an $\mathcal{S}$-path disjoint from U_0''. Then clearly P is also an $\mathcal{S}''$-path disjoint from U_0'' (it doesn't matter that P might contain u as an internal vertex), and so P contains an edge from some $G[U_i'']$ $(i \neq 0)$. This shows that $U_0'', \ldots, U_n''$ is a feasible partition w.r.t. $\mathcal{S}$. Moreover, $S'' = S \cup \{u\}$, the edge set remain unchanged and, thus, $B_i'' \supseteq B_i$ for all $i \in [n]$. (Obviously, B_i'' denotes the set of vertices of U_i that belong to S'' or have (at least) one neighbour in $V \backslash (U_0 \cup U_i)$.) It follows that the minimum in (1) w.r.t. $\mathcal{S}''$ is not less than μ.

Thus, since $(E, \mathcal{S}')$ isn't a counterexample, there exists a collection $\mathcal{L}$ of μ disjoint $\mathcal{S}''$-paths. Choose $\mathcal{L}$ such that no internal vertex of any path in $\mathcal{L}$ belongs to S'' and such that the paths of $\mathcal{L}$ uses the minimum number of edges not used by $\mathcal{P}$.

Fact 2.11. *(i) No path of $\mathcal{L}$ has an internal vertex in S''*

(ii) Choose $\mathcal{L}$ such that the paths of $\mathcal{L}$ uses the minimum number of edges of $E - \bigcup_{H \in \mathcal{P}} E(H)$.

If no path of $\mathcal{L}$ has u as an end vertex, then $\mathcal{L}$ is a collection of μ disjoint $\mathcal{S}$-paths, and we are done. Hence, we may assume that some path $Q_0 \in \mathcal{L}$ has u as an end vertex. It follows from the definition of $\mathcal{S}''$ [see (9)] that every path of $\mathcal{L} \backslash \{Q_0\}$ is an $\mathcal{S}$-path.

Fact 2.12. *Every path of $\mathcal{L} \backslash \{Q_0\}$ is an $\mathcal{S}$-path.*

We have P_0 in $\mathcal{P}$, and u is an internal vertex of P_0. Hence u is not an end vertex of any path in $\mathcal{P}$. But u is an end vertex of Q_0 in $\mathcal{L}$.

As $|\mathcal{P}| = \mu = |\mathcal{L}|$, there must exist an end vertex v of some path $P \in \mathcal{P}$ which is *not* an end vertex of any path in $\mathcal{L}$.

Fact 2.13. *The path $P \in \mathcal{P}$ has an end vertex v which is not an end vertex of any path in $\mathcal{L}$, in particular, $v \neq u$.*

Claim 2.14. *If P does not intersect any path in $\mathcal{L}$, then there exist μ disjoint $\mathcal{S}$-paths.*

Proof of Claim 2.14. If $P = P_0$, then P and $Q_0 \in \mathcal{L}$ intersect in u; a contradiction. Hence $P \neq P_0$, and so, by Fact 2.10, P is an $\mathcal{S}$-path. Moreover, by Fact 2.12, every path of $\mathcal{L}\backslash\{Q_0\}$ is an $\mathcal{S}$-path. Now $(\mathcal{L}\backslash\{Q_0\}) \cup \{P\}$ is a set of μ disjoint $\mathcal{S}$-paths, and we got what we needed. ☐

According to Claim 2.14, we may assume that P intersects at least one path in $\mathcal{L}$.

Fact 2.15. *The path P intersects at least one path in $\mathcal{L}$.*

Thus, when following P starting at v, there is a first vertex w that is one some path $\mathcal{L}$, say, on $Q \in \mathcal{L}$.
(The vertex w can't be identical to v, and here is the reason why: Suppose $v = w \in Q \in \mathcal{L}$. The vertex v is not an end vertex of any path in $\mathcal{L}$, and so v must be an internal vertex of Q. But, since v is an end vertex of the $\mathcal{S}'$-path P, it follows that $v \in S'$. Since $S' \subseteq S''$, we have shown that Q contains an internal vertex $v \in S''$, which contradicts Fact 2.11 (i).)

Let U denote the set of $\mathcal{S}''$ containing v. (Such a set exists, since $v \in S' \subseteq S''$) Let y and z denote the end vertices of the path Q.

For any end vertex x of Q, let Q^x denote the $x - w$ part of Q. Let P^v denote the $v - w$ part of P.

Claim 2.16. *We may assume that for any end vertex x of Q,*

(A) Q^x is part of P, or

(B) the other end of Q (that is, the end of Q different from x) belongs to U.

Proof of Claim 2.16. By symmetry, we may assume $x = y$. Assume that (A) does not hold, that is, Q^y is not part of P. Suppose (B) does not hold, that is, $z \notin U$. Then z is contained in some $\mathcal{S}''$ distinct from U, say R.

Then the situation looks like the one depicted in Figure 4. Then $F := P^v \cup Q^z$ is an $\mathcal{S}''$-path between R and U. Now $(\mathcal{L}\backslash\{Q\}) \cup \{F\}$ is a set of μ disjoint $\mathcal{S}''$-paths with less edges in $E - \bigcup_{H \in \mathcal{P}} E(H)$. (Since Q^y is not part of P at least the first edge of Q_y incident with w is not contained in any path

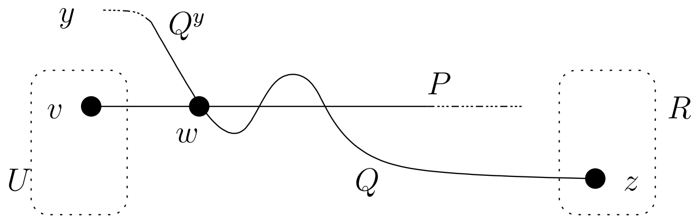

Figure 4: The situation in the argument of Claim 2.16.

$H \in \mathcal{P}$.) This contradicts Fact 2.11 (ii). Hence z must belong to U and so (B) hold. □

Since Q is an $\mathcal{S}''$-path and U is an $\mathcal{S}''$-set, we cannot have both end vertices y and z in U. By symmetry, we may assume $y \notin U$. Now it follows from Claim 2.16 that $Q^z \subseteq P$.

Claim 2.17. *(i)*

$$U = T \cup \{u\} \text{ and } P = P_0.$$

(ii)

$$Q = Q_0 \text{ and } u = z.$$

Proof of Claim 2.17. Suppose[3] Q^y is part of P. Then $Q \subseteq P$.

If $u \in Q$, then $Q = Q_0$, since u is on Q_0. Moreover, $u \in P_0$ and $u \in Q \subseteq P$, which implies $P = P_0$. Recall, that P_0 is a path starting at $s \in T$, going through u and ending at some vertex of $T \backslash \{s\}$. Thus, the end vertices of $P = P_0$ are in $T \cup \{u\}$, which is a set of $\mathcal{S}''$. But, by definition, U is the $\mathcal{S}''$-set containing the end vertex v of P, and so $U = T \cup \{u\}$. Now, y and z denotes the end vertices of Q, u is an end vertex of Q_0, $Q = Q_0$, $u \in \{u\} \cup T = U$ and $y \notin U$. Hence it must be the case that z is identical to u. In this case, we have established both (i) and (ii) of Claim 2.17.

Hence we may assume $u \notin Q$. If Q is a *proper* subset of P, then (at least) one of the end vertices of Q, let's just for now called it a, would be an internal vertex of P. Now, since Q is an $\mathcal{S}''$-path and $u \notin Q$, we get $a \in S'' \backslash \{u\} = S'$, that is, the $\mathcal{S}'$-path P has an internal vertex in S'. This contradicts Fact 2.9, and so we must have $Q = P$. However, the vertex v is an end vertex of P, but v is not an end vertex of any path in $\mathcal{L}$, see Fact 2.13. Since $Q \in \mathcal{L}$, we

[3] I feel that the proof of Claim 2.17 is a bit messy! Can you find a shorter argument?

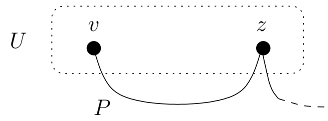

Figure 5: Illustration for the proof of Claim 2.17.

have obtained a contradiction, which implies that Q^y is *not* a part of P.

Since Q^y is not a part of P, it follows from Claim 2.16 that z must belong to U. Since Q^z is part of P, we are in situation as depicted in Figure 5.

Observe, that U cannot be in $\mathcal{S}'$, since then the $\mathcal{S}'$-path P of $\mathcal{P}$ would contain z as an internal vertex and $z \in S'$, which contradicts Fact 2.9. Since $\mathcal{U} \in \mathcal{S}''$, it follows from (8) and (9) that U is identical to $T \cup \{u\}$.

Suppose that not both v and z are in T. We know that $v, z \in U = T \cup \{u\}$ and $v \neq u$ (see Fact 2.13). Hence $v \in T$ and $z = u$. Now $u \in Q^z \subseteq P$ and $u \in P_0$. Hence $P = P_0$. Similarly, $u \in Q$ and $u \in Q_0$ and $Q = Q_0$. Thus, we have established both (i) and (ii).

Suppose that both v and z are in T. Recall, that the path $P \in \mathcal{P}$ is an $\mathcal{S}'$-path with no internal vertex in S'. This is not possible if $z \neq s$ and $v \neq s$. Hence P must contain the vertex s, and, since P_0 is the unique path in $\mathcal{P}$ containing s. Thus, $P = P_0$, and we have established (i).

Let v' denote the end vertex of P distinct from v.

The vertex w is defined to be the first vertex of P belonging to a path of $\mathcal{L}$. Since u is an end vertex of $Q_0 \in \mathcal{L}$, it follows that w must be a vertex on the $u - v$ subpath of P. We have $Q^z \subseteq P$, $z \in U$ and $v \neq w$. Since w was the first vertex of P belonging to Q, it must be the case that Q^z is a path starting at w and leading in the direction of v' along P. It also means that z must be either u or v'. In any case, u ends up being a vertex of Q, and so we must have $Q = Q_0$. This establishes (ii). □

Now we are in a situation as depicted in Figure 6. The $\mathcal{S}''$-path $Q \in \mathcal{L}$ has end vertices $z = u$ and $y \notin U$. Let R denote the $\mathcal{S}''$-set containing y.

Let F denote the path starting at y, going along Q to w and then going along P to v.

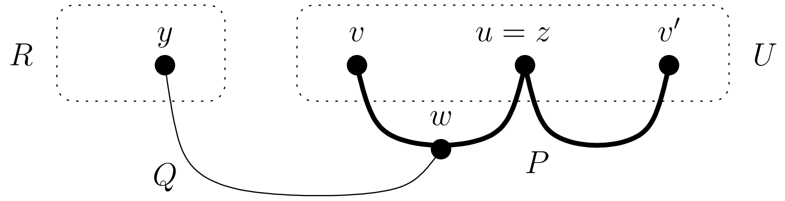

Figure 6: The McDonald's configuration – illustration for the final argument in Part II.

(1) F is an $\mathcal{S}$-path going from $y \in R \in S''\backslash\{U\} = \mathcal{S}\backslash\{T\}$ to $v \in T \in \mathcal{S}$.

(2) F is disjoint from all paths of $\mathcal{L}\backslash\{Q\}$.

(3) Each path of $\mathcal{L}\backslash\{Q\}$ is an $\mathcal{S}$-path (follows from Fact 2.12, since $Q = Q_0$).

From (1-3) we conclude that $(\mathcal{L}\backslash\{Q\}) \cup \{F\}$ is a collection of μ disjoint $\mathcal{S}$-paths. This, finally, completes the proof.

□

3 The Tutte-Berge Formula

This section contains a proof of the Tutte-Berge formula which relies on Mader's $\mathcal{S}$-Paths Theorem.

Theorem 3.1 (Tutte [7], Berge [2]). *For any graph G, the cardinality of a maximum matching in G is equal to*

$$\frac{1}{2} \min_{A \subseteq V(G)} \left(|V(G)| + |A| - o(G - A)\right). \tag{10}$$

Let $\alpha'(G)$ denote the matching number of G, that is, the cardinality of a maximum matching in G.

Proof. We shall first see that $\alpha'(G)$ is bounded above by the value in (10). Let A denote a subset of $V(G)$, and let M denote a matching in G. Let U denote the set of vertices not covered by any edge of M. We wish to establish the following inequality.

$$|U| \geq o(G - A) - |A|. \tag{11}$$

If $o(G-A)=0$, then (11) obviously holds. Otherwise, let H denote an odd component of $G-A$. For M to cover every vertex of $V(H)$, at least one vertex of $V(H)$ must be matched by M to some vertex of A. Obviously, no more than $|A|$ vertices of $V(G-A)$ can be matched to vertices of A, and therefore at least $o(G-A)-|A|$ vertices in the odd components of $G-A$ are not matched by M, in particular, (11) holds. Moreover,

$$|M| \leq \frac{|V(G)-U|}{2} \leq \frac{1}{2}\left(|V(G)|+|A|-o(G-A)\right). \tag{12}$$

Since (12) holds for every subset A of $V(G)$ and matching M in G, we find that $\alpha'(G)$ is indeed bounded above the value in (10).

Now for the converse. Let $\mathcal{S}$ denote the set of all 1-element subsets of $V(G)$. Then μ (the minimum occurring in Mader's $\mathcal{S}$-Paths Theorem) is equal to the matching number $\alpha'(G)$ of G. Let A denote any subset of $V(G)$. We just need to show that μ is at least $(|V(G)|+|A|-o(G-A))$.

Let $U_0 := A$ and let $U_1,\ldots,U_n$ denote the components of $G-U_0$. (We have $n=0$, if $A=V(G)$.)

By considering the parity of the sets U_i for $i\in[n]$, we obtain

$$\sum_{i=1}^{n}\left\lfloor\frac{|U_i|}{2}\right\rfloor = \frac{1}{2}|V(G)-U_0| - \frac{1}{2}o(G-U_0), \tag{13}$$

which implies

$$\frac{1}{2}\left(|V(G)|+|U_0|-o(G-U_0)\right) = |U_0| + \sum_{i=1}^{n}\left\lfloor\frac{|U_i|}{2}\right\rfloor. \tag{14}$$

Now we just need to make sure that the right-hand side of (14) is at least μ. The follows from the following observations.

(i) The vertex sets of the components $U_0,\ldots,U_n$ form a partition of V.

(ii) Each $\mathcal{S}$-path disjoint from U_0 traverses some edge spanned by some U_i (simply because $U_1,\ldots,U_n$ are the components of $G-U_0$).

(iii) $B_i = U_i$, since $S=V(G)$.

Observations (i-ii) imply that $U_0,\ldots,U_n$ is a feasible partition w.r.t. G and $\mathcal{S}$, and so, since $B_i=U_i$, the right-hand side of (14) is at least μ. Since $\mu=\alpha'(G)$ and the subset A of G was chosen arbitrarily we obtain

$$\alpha'(G) \geq \frac{1}{2}\min_{A\subseteq V(G)}\left(|V(G)|+|A|-o(G-A)\right).$$

This completes the derivation of the Tutte-Berge formula. $\square$

For the sake of completeness I include a short self-contained induction proof of Tutte-Berge formula[4].

Proof of Theorem 3.1. First, let's, again[5], see that the value in (10) is an upper bound of the matching number $\alpha'(G)$, we have, for each $U \subseteq V(G)$,

$$\alpha'(G) \leq |U| + \alpha'(G-U) \leq |U| + \frac{1}{2}\left(|V\backslash U| - o(G-U)\right)$$
$$= \frac{1}{2}\left(|V| + |U| - o(G-U)\right).$$

The reverse inequality is proved by induction on the order of $n(G)$. If $n(G) \leq 1$, then the desired inequality hold. We can assume that G is connected, as otherwise we can apply induction to the components of G.

Firstly, assume that there exists a vertex v covered by all maximum matchings. Let M denote a maximum matching of $G - v$. Then M is *not* a maximum matching of G (since M doesn't cover v), and therefore $\alpha'(G-v) = |M| < \alpha'(G)$. Let M' denote a maximum matching of G and let e denote the edge of M' covering the vertex v. Now $M'\backslash\{e\}$ is a matching of $G - v$ and therefore $\alpha'(G-v) \geq |M'\backslash\{e\}| = \alpha'(G) - 1$. Together these two inequalities imply $\alpha'(G-v) = \alpha'(G) - 1$. Now the induction hypothesis applied to the graph $G - v$ implies that there exists a subset U' of $V(G)\backslash\{v\}$ such that

$$\alpha'(G-v) = \frac{1}{2}\left(|V\backslash\{v\}| + |U'| - o(G-v-U')\right). \tag{15}$$

Then $U := U' \cup \{v\}$ gives the desired inequality.

Secondly, assume that there is no such vertex v. This obviously implies $\alpha'(G) \leq n(G)/2$. *If* G has a matching of size $(n(G)-1)/2$, then $n(G)$ is odd, and, since G is connected, $o(G) = 1$. Now, taking $U := \emptyset$ gives the desired inequality. Thus, we need only show that G has a matching of size $(n(G)-1)/2$.

Suppose to the contrary that every maximum matching M of G misses at least two distinct vertices u and v. Among all such M, u and v, choose

[4]The proof stated here is an almost verbatim copy of a proof given by Schrijver [1].

[5]This argument is even short than the one presented in the previous proof of Theorem 3.1.

them such that the distance $\text{dist}(u, v)$ is as small as possible.

By the maximality of the matching M, $\text{dist}(u, v) \geq 2$. Let t denote an internal vertex on a shortest $u - v$ path. (Obviously, t is covered by M.) By assumption, there exists a maximum matching of N of G missing t. Choose such a matching N with $|M \cap N|$ maximal.

By the minimality of $\text{dist}(u, v)$, N covers both u and v. Hence, as M and N cover the same number of vertices, there exists a vertex $x \neq t$ covered by M but *not* by N. Let $e = xy$ denote the edge of M containing x. The vertex y must be covered by some edge f in N, otherwise $N \cup \{e\}$ would be a matching of G. This implies $y \neq t$. Obviously, f is not in M, since f covers y and $e = xy \in M$.

Now $N' := (N \backslash \{f\}) \cup \{e\}$ is a maximum matching of G which misses t, and $|M \cap N'| > |M \cap N|$. This contradicts our choice of N. Hence, no maximum matching of G misses more than one vertex, and therefore, since $\alpha'(G) < n(G)/2$, indeed, $\alpha'(G) = (n(G) - 1)/2$. □

As a corollary to Theorem 3.1 (the Tutte-Berge formula) we immediately obtain the following classic result, which is known as *Tutte's 1-factor*[6] *Theorem*, *Tutte's Factorization Theorem*, or, simply, *Tutte's Theorem*.

Corollary 3.2 (Tutte [7]). *A graph G has a perfect matching if and only if $o(G - A) \leq |A|$ for all subsets A of $V(G)$.*

As an application of Tutte's 1-factor Theorem we give a short proof of one of the very first theorems of graph theory, *Petersen's Theorem*[7].

Corollary 3.3 (Petersen [5]). *Every bridgeless cubic graph has a 1-factor.*

A *bridge* of a graph G is an edge of G which separates two distinct vertices of the same component of G. Thus, the bridges of G are precisely those edges that do not lie on any cycle of G.

Proof of Corollary 3.3. Let G denote a bridgeless cubic graph, and let A denote some arbitrary subset of $V(G)$. Suppose C is an odd component of $G - A$. Given disjoint subsets X and Y of $V(G)$, let $E(X, Y)$ denote the set of edges connecting vertices of X to vertices of Y. Then

$$\sum_{v \in V(C)} \deg_G(v) = 2|E(C)| + |E(V(C), A)| \tag{16}$$

[6] A *1-factor* of a graph G is a perfect matching of G.

[7] Of course, these notes wouldn't be complete without a result of the famous Danish pioneer of graph theory, Julius Petersen (1839-1910).

Since G is cubic, and C is an odd component of G, the sum $\sum_{v \in V(C)} \deg_G(v)$ must equal an odd number. Thus, (16) implies that $|E(V(C), A)|$ is an odd number, in particular, since G is bridgeless, $|E(V(C), A)| \geq 3$. Thus,

$$3o(G - A) \leq |E(V(G) \backslash A, A)|.$$

On the other hand, since G is cubic,

$$|E(V(G) \backslash A, A)| \leq 3|A|,$$

Altogether, we obtain $o(G - A) \leq |A|$, that is, G satisfies Tutte's condition and therefore, according to Corollary 3.2 (Tutte's Theorem), G has a 1-factor. □

References

[1] URL `http://homepages.cwi.nl/~lex`.

[2] C. Berge. Sur le couplage maximum d'un graphe. *C. R. Acad. Sci. Paris*, 247:258–259, 1958.

[3] T. Gallai. Maximum-minimum Sätze und verallgemeinerte Faktoren von Graphen. *Acta Math. Acad. Sci. Hungar.*, 12:131–173, 1961.

[4] W. Mader. Über die Maximalzahl kreuzungsfreier H-Wege. *Arch. Math. (Basel)*, 31(4):387–402, 1978/79.

[5] J. Petersen. Die Theorie der regulären graphs. *Acta Math.*, 15(1):193–220, 1891.

[6] A. Schrijver. A short proof of Mader's $\mathcal{S}$-paths theorem. *J. Comb. Theory, Ser. B*, 82(2):319–321, 2001.

[7] W. T. Tutte. The factorization of linear graphs. *J. London Math. Soc.*, 22:107–111, 1947.