Notes on the Probabilistic Method

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Preface

This manuscript contains my personal and informal notes on the probabilistic method as it is applied in graph theory. It hasn't been properly proof-read, and it is by no means intended for wider distribution. For an introduction to the probabilistic method, I refer the reader to the text-books mentioned in the bibliography. I'm very grateful for the assistance and advice I have received from Marco Chiarandini, Troels Steenstrup, Matthias Kriesell, Mike Molloy and Bjarne Toft. They are, of course, by no means responsible for the present manuscript.

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Chapter 1

Introduction

1.1 Elementary probability theory

A random variable is a function assigning a real number to each element of a probability space. For any non-negative integer-valued random variable X, the expected value of X, denoted $E(X)$, is the number defined by the sum

$$\sum_{i \geq 0} \big(i \cdot P(\{X = i\}) \big)$$

Given a random variable X, the *pigeonhole property of the expectation* is the fact that there exists an element of the probability space for which the value of X is as large (or as small as) $E(X)$. As usual, we write $X \sim F(\phi)$ to indicate that the random variable X follows a distribution F which may depend on some parameter ϕ. We let $\text{bin}(n, p)$ denote the binomial distribution. Recall, that $X \sim \text{bin}(n, p)$ ($n \in \mathbb{N}$ and $p \in [0, 1]$) if X is the sum of n independent random variables, each equal to 1 with probability p and equal to 0 with probability $1 - p$. If $X \sim \text{bin}(n, p)$, then the expected value $E(X)$ of X is equal to np. Using the definition of the expectation, it is straightforward to deduce Markov's Inequality:

Theorem 1.1.1 (Markov's Inequality)**.** *For any non-negative integer-valued random variable X and positive number t,*

$$P(X \geq t) \leq \frac{E(X)}{t} \tag{1.1}$$

For $t = 0$, we obtain

$$P(X > 0) \leq E(X) \tag{1.2}$$

Theorem 1.1.2 (First Moment Principle)**.** *For any non-negative integer-valued random variable X and positive number t, if $E(X) \leq t$, then $P(X \leq t) > 0$.*

The *First Moment*[1] *Method* is a common expression used for the application of (1.1), (1.2) and the First Moment Principle in a combinatorial setting.

In Section 3.1 we prove that almost every graph is a counterexample to Hajós' Conjecture. In the proof we use the Chernoff Bound, which is a bound on the probability that a binomial random variable is 'far' from its expected value. Given $X \sim \text{bin}(n, p)$ and $t \in [0, np]$, the Chernoff Bound provides a bound on the probability of the event $\{|X - E(X)| > t\}$.

Theorem 1.1.3 (Chernoff Bound)**.** *For any random variable $X \sim \text{bin}(n, p)$ and any number $t \in [0, np]$,*

$$P(|X - np| > t) < 2\exp\left(\frac{-t^2}{3np}\right) \tag{1.3}$$

Recall, that the probability mass function of a binomially distributed random variable $X \sim \text{bin}(n, p)$ is

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{for any integer } k \in [n]$$

(For any positive integer n, we let $[n]$ denote the set $\{1, \dots, n\}$.) For our purposes, the case where the probability p is exactly one half is of special interest. Using the obvious fact that $\binom{n}{k} = \binom{n}{n-k}$ for any integer $k \in [n]$, it is easy to see that

$$P(X - E(X) > t) = P(X - E(X) < -t)$$

Since $P(|X - np| > t) = P(X - E(X) > t) + P(X - E(X) < -t)$, the Chernoff Bound implies

$$P(X - \frac{n}{2} > t) < \exp\left(\frac{-2t^2}{3n}\right) \tag{1.4}$$

in the special case of $p = \frac{1}{2}$.

[1] Recall, that the expected value $E(X)$ of a random variable X also is known as the first moment of X.

Given some set $\mathcal{E} = \{A_1, \ldots, A_n\}$ of events, we say that an event A is *independent of the set of events* $\mathcal{E}$ if for any subset $S \subseteq [n]$ of indices,

$$P\left(A \cap \left(\bigcap_{i \in S} A_i\right)\right) = P(A) \cdot P\left(\bigcap_{i \in S} A_i\right) \tag{1.5}$$

If A is in the set $\mathcal{E}$ and A is independent of $\mathcal{E}$, then either $P(A) = 0$ or $P(A) = 1$. Therefore, let's just require[2] $A \notin \mathcal{E}$. The conditional probability of A given B, denoted $P(A \mid B)$, is

$$\frac{P(A \cap B)}{P(B)}$$

(If $P(B) = 0$, then we define $P(A \mid B)$ to be $P(A)$[3].) Clearly, the event A is independent of the set of events $\mathcal{E}$ if and only if for any subset $S \subseteq [n]$ of indices

$$P(A \mid \bigcap_{i \in S} A_i) = P(A)$$

For convenience, we choose to write the intersection $A_1 \cap A_2 \cap \cdots \cap A_n$ as $A_1 A_2 \ldots A_n$and the union $A_1 \cup A_2 \cup \cdots \cup A_n$ as $A_1 + A_2 + \cdots + A_n$.

Proposition 1.1.4. *Suppose that an event A is independent of a set of events* $\mathcal{E}$. *Then for any* $B_1, \ldots, B_r, C_1, \ldots, C_S \in \mathcal{E}$,

$$P\left(AB_1 \ldots B_r \overline{C_1} \ldots \overline{C_s}\right) = P(A) \cdot P(B_1 \ldots B_r \overline{C_1} \ldots \overline{C_s})$$

Proof. The argument proceeds by induction on the value of s. First the base case $s = 1$. Given an arbitrary subset T of $\mathcal{E}$, let $B := \cap_{D \in T} D$. By definition, we have

$$P\left(ABC_1\right) = P(A) \cdot P(BC_1)$$

which we use in the following calculation. We also use the fact that $P(DE) + P(D\overline{E}) = P(D)$ for any events D and E.

$$\begin{aligned} P(AB\overline{C_1}) &= P(AB) - P(ABC_1) \\ &= P(A)P(B) - P(A)P(BC_1) \\ &= P(A)\left[P(B) - P(BC_1)\right] \\ &= P(A)P(B\overline{C_1}) \end{aligned}$$

[2]This seems easier than always having to consider the case where $A \in \mathcal{E}$.

[3]This may not be completely standard, but at least it is consistent with the definition given by Molloy and Reed [23].

Induction hypothesis: Suppose that for some $j \geq 1$ and any subset T of $\mathcal{E}$,

$$P(AB\overline{C_1}\ldots\overline{C_j}) = P(A) \cdot P(B\overline{C_1}\ldots\overline{C_j})$$

where $B := \cap_{D\in T} D$.

Induction step: Let T denote an arbitrary subset of $\mathcal{E}$, and let $B := \cap_{D\in T} D$. By the complement law,

$$P(AB\overline{C_1}\ldots\overline{C_j}\,\overline{C_{j+1}}) = P(AB\overline{C_1}\ldots\overline{C_j}) - P(AB\overline{C_1}\ldots\overline{C_j}C_{j+1}) \qquad (1.6)$$

Now the trick is that the induction hypothesis applies to the right-hand side of (1.6). In the second term on the right-hand side of (1.6) we apply the induction hypothesis by letting $B \cup C_{j+1}$ take the place of B. This gives us

$$\begin{aligned} P(AB\overline{C_1}\ldots\overline{C_j}\,\overline{C_{j+1}}) &= P(A)P(B\overline{C_1}\ldots\overline{C_j}) - P(A)P((BC_{j+1})\overline{C_1}\ldots\overline{C_j}) \\ &= P(A)\left[P(B\overline{C_1}\ldots\overline{C_j}) - P((BC_{j+1})\overline{C_1}\ldots\overline{C_j})\right] \\ &= P(A)P(B\overline{C_1}\ldots\overline{C_j}\,\overline{C_{j+1}}) \end{aligned}$$

And so the proof is complete. □

1.2 Basic graph theoretic terminology

For an introduction to graph theory we refer the reader to [9, 11]. In this section we have gathered definitions of some of the graph-theoretic concepts which we shall be using in the following sections.

The path, the cycle and the complete graph on n vertices is denoted P_n, C_n and K_n, respectively. The *length* of a path or a cycle is its number of edges. A *k-colouring* of a graph G is a function φ from the vertex set $V(G)$ of G into a set $\mathcal{C}$ of cardinality k so that $\varphi(u) \neq \varphi(v)$ for every edge $uv \in E(G)$, and a graph is *k-colourable* if it has a k-colouring. The elements of the set $\mathcal{C}$ is referred to as colours, and a vertex $v \in V(G)$ is said to be assigned the colour $\varphi(v)$ by ϕ. The set of vertices S assigned the same colour $c \in \mathcal{C}$ is said to constitute the colour class c. The minimum integer k for which a graph G is k-colourable is called its *chromatic number* of G and it is denoted $\chi(G)$. An *independent set* S of $V(G)$ is a set such that the induced graph $G[S]$ is edge-empty. The maximum integer k for which there exists an independent set S of G of cardinality k is the *independence number* of G and is denoted $\alpha(G)$.

For a vertex v of a graph G, the *(open) neighbourhood* of v in G is denoted $N_G(v)$ while $N_G[v]$ denotes the *closed neighbourhood* $N_G(v)\cup\{v\}$. The *degree* of a vertex v in a graph G is the size of the neighbourhood $N_G(v)$, and it is denoted $\deg_G(v)$ or, simply, $\deg(v)$. The maximum degree and minimum degree in G is denoted $\Delta(G)$ and $\delta(G)$, respectively, while the *average degree* of G (or, to be more precise, the average degree of the vertices of G) is defined as the number

$$d(G) := \frac{1}{n(G)} \sum_{v \in V(G)} \deg_G(v)$$

Obviously, $\delta(G) \leq d(G) \leq \Delta(G)$ and $d(G) = 2m(G)/n(G)$.

The *colouring number*[4] $\mathrm{col}(G)$ of a graph G is the number given by

$$\mathrm{col}(G) = 1 + \max\{\delta(H) \mid H \text{ is an induced subgraph of } G\}.$$

The colouring number $\mathrm{col}(G)$ gives an upper bound on the fewest possible number of colours used by the sequential colouring algorithm when applied to G, in particular, $\chi(G) \leq \mathrm{col}(G) \leq \Delta(G) + 1$ [32, p. 250].

A graph $G = (V, E)$ is said to be *k-list-colourable*, or *k-choosable*, if, for every family $(S_v)_{v \in V}$ with each set S_v having size k, there is a proper vertex colouring φ of G with $\varphi(v) \in S(v)$ for all $v \in V$. The least integer k for which G is k-choosable is the *list-chromatic number* $\chi_\ell(G)$, or the *choice number* $\mathrm{ch}(G)$ of G [11, Sec. 5.4]. For every graph G,

$$\chi(G) \leq \mathrm{ch}(G) \leq \mathrm{col}(G) \leq \Delta(G) + 1$$

A graph H is a *minor* of a graph G if H can be obtained from G by deleting edges and/or vertices and contracting edges. An *H-minor* of G is a minor of G isomorphic to H. Given a graph G, the largest integer k for which G contains a K_k-minor is called the *Hadwiger number* of G, and it is denoted $h(G)$.

In a graph G with at least one cycle, the *girth* of a G, denoted $\mathrm{girth}(G)$, is the length of a shortest cycle. A graph with no cycle is said to have infinite girth.

For $n \geq 0$ and $p \in [0, 1]$, we denote by $\mathcal{G}(n, p)$ the set of random graphs on n vertices where any pair of distinct vertices are adjacent with probability exactly p. Similarly, we may at times let G^p a random graph where any pair of distinct vertices are adjacent with probability exactly p. Given some graph property $\mathcal{P}$, we say that *almost every* graph G^p has the property $\mathcal{P}$ if the

[4]The colouring number is also known as the Szekeres-Wilf number $\mathrm{SW}(G)$ of G. A graph with colouring number k is also said to be *k-degenerate*.

probability that a random graph G^p has property $\mathcal{P}$ goes to 1 as the number of vertices goes to infinity.

Chapter 2

A gentle beginning

We will start out with one of the first result presented by Douglas B. West in his chapter on random graphs in [33]; it is a result by Tibor Szele on the the number of Hamiltonian paths[1] in a tournament. A tournament D on n vertices is an orientation on the complete graph on n vertices, that is, for each pair of vertices $u, v \in V(D)$ either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. There is a simple and elegant combinatorial argument showing that every tournament contains at least one Hamiltonian path. Szele proved, using the pigeonhole property of the expectation, that for every positive integer n there exists a tournament on n vertices which contains at least $\frac{n!}{2^{n-1}} =: f(n)$ Hamiltonian paths. This function f grows very fast, for instance, $f(10) = 7087.5$, and so there is a tournament on 10 vertices which contains, at least, 7088 Hamiltonian paths.

Theorem 2.0.1 (Szele [31]). *For every positive integer n, there is a tournament on n vertices with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.*

Adler, Alon and Ross [1] mention that Szele's proof is considered the first application of the probabilistic method in combinatorics.

In the proof of Theorem 2.0.1 we actually prove that the expected number of Hamiltonian paths in a uniform random tournament on n vertices is $\frac{n!}{2^{n-1}}$.

Proof of Theorem 2.0.1. We consider the uniform random tournament D with vertex set $\{1, \ldots, n\}$. Then, given any pair of distinct vertices $u, v \in V(D)$, we have $(u, v) \in A(D)$ with probability $1/2$. Let X denote the number of Hamiltonian paths in D, and given any permutation π of the sequence $1, 2, \ldots, n$, let X_π denote the indicator variable with is 1 if the sequence

[1] Here we are, obviously, considering *directed* Hamiltonian paths.

$\pi(1), \pi(2), \ldots, \pi(n)$ corresponds to a Hamiltonian path in D. Then

$$X = \sum X_\pi$$

where the sum is taken over all permutations π of the sequence $1, 2, \ldots, n$. Now, given any such permutation π, the probability of X_π occurring is precisely $\frac{1}{2^{n-1}}$ (right?). And, since there are $n!$ such permutations, we get, by the linearity of expectation,

$$E(X) = \sum P(X_\pi) = n! \times \frac{1}{2^{n-1}}$$

This implies the existence of at least one tournament with, at least, $\frac{n!}{2^{n-1}}$ Hamiltonian paths. □

More than 50 years later, Adler, Alon and Ross [1] obtained the following asymptotic improvement: Let $P(n)$ denote maximum number of Hamiltonian paths in a tournament on n vertices. Then

$$P(n) \geq (e - o(1))\frac{n!}{2^{n-1}}$$

where the term $o(1)$ tends to 0 as n tends to infinity.

As for upper bounds on $P(n)$, Szele [31] proved $P(n) \in O(n!/2^{3n/4})$, which was improved by Alon [2] who proved $P(n) \in O(n^{3/2}n!/2^n)$. Adler, Alon and Ross [1] asked whether $P(n) \in \Theta(n!/2^n)$, which still seems to be an open question.

We proceed our introduction to the probabilistic method by proving an upper bound on the domination number. Recall, that a *dominating set* of a graph G is a set $S \subseteq V(G)$ such that each vertex of $V(G)\backslash S$ has a neighbour in S. The *domination number* of G is the cardinality of a minimum dominating set of G, and it is denoted $\gamma(G)$. (Obviously, for any graph G of order n and maximum degree Δ, $\gamma(G) \geq n/(\Delta + 1)$.)

Theorem 2.0.2. *For any graph G with order n and minimum degree δ,*

$$\gamma(G) \leq n \cdot \frac{1 + \ln(\delta + 1)}{\delta + 1} \tag{2.1}$$

This result seems first to have been proved by Arnautov [6] and Payan [26] using a combinatorial argument. (A short combinatorial argument may be found in [34, p.117]. Here we present an elegant probabilistic argument.

Proof. (Alon [4, p.6]) Let G denote an arbitrary graph of order n and minimum degree[2] δ. We are going to construct a set $S \subseteq V(G)$ by including each vertex $v \in V(G)$ into S independently with probability p. The trick is to choose the right value for p. Taking a look in the crystal ball we see that the right value[3] for p is $\ln(\delta+1)/(\delta+1)$.

Given S, let $T := V(G)\backslash N[S]$. Then $S \cup T$ is a dominating set of G. Let's calculate the expected size of $S \cup T$.

Since each vertex of G appears in S with probability p, linearity of expectation yields[4] $E(|S|) = np$.

We have $v \in T$ if and only if v and all its neighbours are not in S. This event has probability $(1-p)^{\deg(v)+1} \leq (1-p)^{\delta+1}$. Hence $E(|T|) \leq n(1-p)^{\delta+1}$. Since $1 - x < e^{-x}$ for every positive number x, we obtain $E(|T|) \leq n\left(e^{-p}\right)^{\delta+1} = ne^{-p(\delta+1)}$. Thus, by the linearity of expectation,

$$E(|S| + |T|) \leq np + ne^{-p(\delta+1)} = n \cdot \frac{1 + \ln(\delta+1)}{\delta+1}$$

With this upper bound on the expectation, the pigeonhole property of expectation implies the existence of the desired dominating set of G. This completes the proof. □

So far we have just used the pigeonhole property for expectation. In this next result we'll use the Markov's Inequality (1.2) to show the somewhat surprising fact that, for any arbitrary fixed probability $p \in (0, 1)$, almost every graph G^p has diameter 2 (and hence is connected).

[2]Alon [4, p.6] requires the graph to have minimum degree at least two, but that doesn't seem necessary, or am I overlooking something?

[3]Obviously, $0 < p < 1$. In our probabilistic arguments we often choose elements at random with probability 1/2 (see, for instance, the proof of Theorem 2.0.1). However, $p = 1/2$ will not work in on this problem, since then the expected size of S would be $n/2$, which would only imply the existence of a dominating set of size at most $n/2$. But

$$f(n, \delta) := n \cdot \frac{1 + \ln(\delta+1)}{\delta+1}$$

is strictly less than $n/2$ for $\delta \geq 5$; moreover, $f(n, \delta)/n \to 0$ as $\delta \to \infty$.

[4]This is, of course, a slight abuse of notation to write $E(|S|)$, but it is obviously meant to indicate the expected value of the random variable expressing the size of the random set S.

Theorem 2.0.3. *For every fixed number $p \in (0,1)$, almost every graph G^p has diameter 2.*

In 1959, Gilbert [16] published a paper proving that almost every graph is connected. The stronger result that almost every graph has diameter 2 seems to have been first published by Blass and Harary [7].

Proof of Theorem 2.0.3. Suppose we are given some graph G on n vertices, where any pair of distinct vertices are adjacent with probabilities p independently any other adjacency in the graph. Let X denote the random variable expressing the number of 2-sets $\{u,v\} \subseteq 2^{V(G)}$, where u and v has no common neighbour in G.

According to Markov's Inequality, we only need to prove that $E(X) \to 0$ for $n \to \infty$, since this will imply $P(X > 0) \to 0$ for $n \to \infty$.

For each such 2-set $\{u,v\}$, let $X_{\{u,v\}}$ denote the indicator variable which is 1 if u and v have no common neighbour and otherwise 0.

By linearity of expectation, we only need to calculate the expected value of $X_{\{u,v\}}$ for each such 2-set. Since $E(X_{\{u,v\}}) = P(X_{\{u,v\}})$, let's calculate $P(X_{\{u,v\}})$.

For any vertex $w \in V(G)\backslash\{u,v\}$,

$$P\,(w \text{ not adj. to both } u \text{ and } v) = 1 - P\,(w \text{ adj. to both } u \text{ and } v\ ) = 1 - p^2$$

Hence $P(X_{\{u,v\}} = 0) = (1-p^2)^{n-2}$, and

$$\begin{aligned} E(X) &= \sum_{\{u,v\}} E(X_{\{u,v\}}) = \sum_{\{u,v\}} P(X_{\{u,v\}}) \\ &= \binom{n}{2} P(X_{\{u,v\}}) = \binom{n}{2}(1-p^2)^{n-2} \\ &\leq cn^2 s^n = c\frac{n^2}{t^n} \end{aligned}$$

where c and s,t are positive constants, in particular, $1/t = s \in (0,1)$. Since $t > 1$, we have $n^2 \in o(t^n)$ (see, for instance, [10, p. 52]), and so $E(X) \to 0$ for $n \to \infty$. This, almost, completes the proof. We have showed that the probability of $\text{diam}(G) \leq 2$ goes to 1; what we need to prove is that the probability of $\text{diam}(G) = 2$ goes to 1. However, the probability of $\text{diam}(G) \leq 1$ obviously goes to 0, since $p < 1$, and so the probability of $\text{diam}(G) = 2$ goes to 1. □

The above result may also be deduced from the following result.

Theorem 2.0.4 (Blass and Harary [7]). *Let s and t denote arbitrary, but fixed nonnegative integers, and let p denote a fixed number in the interval $(0, 1)$.*

Then almost every graph G^p has the property that:

for any pair of disjoint subsets $S, T \subseteq V(G^p)$ of size $\leq s$ and $\leq t$, respectively, some vertex of G^p is adjacent to every vertex of S and to no vertex of T.

Proof. (Palmer [25, p.14]) As usual, we let n denote the number of vertices of G^p. We may suppose that n is much bigger than both $s + t$. A pair of disjoint subsets $S, T \subseteq V(G^p)$ of size $\leq s$ and $\leq t$, respectively, is *bad pair* if no vertex of G^p is adjacent to every of S and no vertex of T. Let X denote the random variable expressing the number of such bad pairs S and T with $|S| = s$ and $|T| = t$. (If G^p contains a bad pair S' and T' with $|S'| < s$ or $|T'| < t$, then, since $s + t >> n$, the bad pair S' and T' may be extended in the obvious way to a bad pair S and T with $|S| = s$ and $|T| = t$. Hence it suffices to show that $E(X) \to 0$ for $n \to \infty$.) The expected value of X is easily seen to be

$$E(X) = \binom{n}{s, t, n-s-t} \left(1 - p^s(1-p)^t\right)^{n-s-t}$$

(Taking $s = 2$ and $t = 0$, we get the expression for $E(X)$ in the proof of Theorem 2.0.3.) Again[5], we obtain $E(X) \to 0$ for $n \to \infty$, and so Markov's Inequality implies that the probability of a random graph contains a bad pair goes to zero as the number of vertices n tends to infinity. □

Here is the trick to obtain Theorem 2.0.3 from Theorem 2.0.4, let G^p denote a random graph and consider any pair of distinct vertices u and v of G^p.

[5]Here are some boring details:

$$\begin{aligned} E(X) &= \frac{n!}{s!t!(n-s-t)!} \left(1 - p^s(1-p)^t\right)^{n-s-t} \\ &< \frac{n^{s+t}}{s!t!} x^{n-s-t} \\ &< c\frac{n^{s+t}}{y^n} \end{aligned}$$

where c and $1/y := x$ are positive constants; in particular, $y > 1$ and so $n^{s+t} \in o(y^n)$, which implies $E(X) \to 0$ for $n \to \infty$.

Let $S = \{u, v\}$ and $T = \emptyset$. Almost surely there is a vertex x in G^p adjacent to all vertices of S, that is, to both u and v, and therefore, $\operatorname{dist}(u, v; G^p) \leq 2$.

Corollary 2.0.5 (Blass and Harary [7])**.** *For any fixed positive integer k, almost every graph G^p is k-connected.*

Proof. Let $s := 2$ and $t := k - 1$. Suppose G^p is a random graph of order $n > s + t$. Let C denote an arbitrary subset of $V(G)$ of size at most t. We intend to show that, almost surely, C isn't a cut-set of G^p. According to Theorem 2.0.4, almost surely G^p doesn't contain a bad pair S and T, where $|S| = s$ and $|T| \leq t$. In particular, $T := C$ and $S = \{u, v\}$, where u and v denote arbitrary distinct vertices[6] of $V(G)\backslash T$, almost surely isn't a bad pair, and so almost surely there exist some vertex $x \in V(G^p)\backslash(S \cup T)$ adjacent to every vertex of S. In particular, $T = C$ almost surely isn't a cut-set of G^p. This implies the desired result. □

Corollary 2.0.6. *Let H denote an arbitrary, but fixed graph. Then almost every graph G^p contains an induced subgraph isomorphic to H.*

The following results are found in [11, Chap. 11].

Lemma 2.0.7. *For all integers n, k with $n \geq k \geq 2$, the probability that a random graph $G \in \mathcal{G}(n, p)$ has a set of k independent vertices is at most*

$$\binom{n}{k}(1-p)^{\binom{k}{2}}$$

Proof. Let $\mathcal{I}$ denote the set of k-set subsets of $V(G)$. The probability that any fixed k-set $U \in \mathcal{I}$ is an independent set of G is $(1-p)^{\binom{k}{2}}$. Thus,

$$\begin{aligned} P(\alpha(G) \geq k) &= P\left(\bigcup_{S \in \mathcal{I}} \{S \text{ is an independent set of } G\}\right) \\ &\leq \sum_{S \in \mathcal{I}} P\left(S \text{ is an independent set of } G\right) = \binom{n}{k}(1-p)^{\binom{k}{2}}, \end{aligned}$$

where we used the subadditivity of probabilities. □

[6]Here we use $n > s + t$.

Similarly, the probability that a random graph $G \in \mathcal{G}(n,p)$ contains the complete k-graph is at most

$$\binom{n}{k} p^{\binom{k}{2}}$$

If, for some fixed n,k, and p, the probabilities $P(\alpha(G) \geq k)$ and $P(\omega(G) \geq k$ add to *strictly* less than 1, then $\mathcal{G}(n,p)$ contains a graph G which has neither a K_k or an induced $\overline{K_k}$. In such a case the Ramsey number $R(k)$ is greater than n. (Recall, that the Ramsey number $R(k)$ is the smallest integer j such than any graph on at least j vertices has K_k or $\overline{K_k}$ as an induced subgraph.

Theorem 2.0.8 (Erdős [12]). *For every integer $k \geq 3$, the Ramsey number $R(k)$ is greater than $2^{k/2}$.*

According to Diestel [11, p. 314], Erdős' proof of Theorem 2.0.8 is one of the origins on the Probabilistic Method.

Proof of Theorem 2.0.8. For $k = 3$, we trivially have $R(3) > 3 > 2^{3/2}$. Hence we may assume $k \geq 4$, and for such k we have $k! > 2^k$, which we shall be using in the following computations. Suppose that $n \leq 2^{k/2}$. We wish to show that for any graph $G \in \mathcal{G}(n, \frac{1}{2})$,

$$P\left(\{\alpha(G) \geq k\} \cup \{\omega(G) \geq k\}\right) < 1 \tag{2.2}$$

since this would imply the existence of a graph H on n vertices with $\alpha(H) < k$ and $\omega(H) < k$, and so $R(k) > n$. Since n was chosen arbitrarily subject to the condition $n \leq 2^{k/2}$, we would obtain $R(k) > 2^{k/2}$.

According to Lemma 2.0.7,

$$P(\alpha(G) \geq k), P(\omega(G) \geq k) \leq \binom{n}{k} \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

Thus, using the subadditivity of properties, we obtain

$$\begin{aligned}
P\left(\{\alpha(G) \geq k\} \cup \{\omega(G) \geq k\}\right) &\leq P(\alpha(G) \geq k) + P(\omega(G) \geq k) \\
&\leq 2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&\leq 2\frac{n!}{(n-k)!k!}\frac{1}{2^{k(k-1)/2}} \\
&\leq 2\frac{n^k}{k!}\frac{1}{2^{k(k-1)/2}} \\
&< 2\frac{n^k}{2^k}\frac{1}{2^{k(k-1)/2}} \\
&\leq 2\frac{(2^{k/2})^k}{2^k}\frac{1}{2^{k(k-1)/2}} \\
&\leq 2\frac{2^{k^2/2}}{2^k}\frac{1}{2^{k^2/2}2^{-k/2}} \\
&= 2\frac{1}{\sqrt{2}^k} \leq \frac{1}{2}.
\end{aligned}$$

Thus, (2.2) is established and the proof is complete. □

We have seen that it is indeed possible to obtain deep results just using the First Moment Principle; for another example of this, see Theorem 3.6.1 on total colouring in Section 3.6.

Chapter 3

More advanced techniques

3.1 Hajós' Conjecture and complete subdivisions

Hajós conjectured that Hadwiger's Conjecture could be strengthened to subdivisions rather than minors: if $\chi(G) \geq k$, then G contains a subdivision of the complete k-graph. Hajós' Conjecture is true for $k \leq 4$, but, as Catlin showed, false for $k \geq 7$; the cases $k = 5$ and $k = 6$ remain open. Erdős and Fajtlowicz proved that almost every graph is a counterexample to Hajós' Conjecture. For more information on Hajós' Conjecture see Thomassen [**?**] and Jensen and Toft [20, Chap. 6].

The expression that *almost every graph* has a certain property $\mathcal{P}$ means that the probability that a uniform random graph on n vertices has property $\mathcal{P}$ tends to 1 as n tends to infinity.

In the remainder of this section, which is based on Molloy and Reed [23, Chap. 5], we let log denote the logarithm with base 2.

Theorem 3.1.1. *For $n \geq 9$, the probability that a uniform random graph G on n vertices has $\chi(G) \geq \frac{n}{2\log(n)}$ and no $K_{8\sqrt{n}}$-subdivision is at least*

$$1 - \frac{1}{n} - \frac{1}{e^{\frac{n}{4}}}$$

Before we proceed to prove Theorem 3.1.1, we deduce an immediate corollary.

Corollary 3.1.2. *Almost every graph is a counterexample to Hajós' Conjecture.*

Proof. For $n > 70.000$, we have $8\sqrt{n} < \frac{n}{2\log(n)}$. Now, according to Theorem 3.1.1, the probability that a uniform random graph G on n vertices has no $K_{\chi(G)}$-subdivision is at least

$$1 - \frac{1}{n} - \frac{1}{e^{-\frac{n}{4}}}$$

which tends to 1 as n tends to infinity. (For $n \geq 70.000$ the probability of a uniform random graph on n vertices being a counterexample is at least 0.999986) This shows that almost every graph is a counterexample to Hajós Conjecture. □

The proof of Theorem 3.1.1 doesn't say anything about the presence or absence of complete subdivisions in graphs on a smaller number of vertices - However, Catlin's counterexample $C_5[K_3]$ is a graph on just 15 vertices.

Proof of Theorem 3.1.1. Let G denote a uniform random graph on n vertices, that is, given any pair of vertices u and v in $V(G)$, the probability of uv being present in G is 1/2.

Firstly, we'll show that $\chi(G) < \frac{n}{2\log(n)}$ with low probability. Since $n/\alpha(G) \leq \chi(G)$ for any graph of order n, we find that the event $\chi(G) < \frac{n}{2\log(n)}$ implies the event $2\log(n) < \alpha(G)$ which implies the event $\lceil 2\log(n) \rceil \leq \alpha(G)$. Thus,

$$P\left(\chi(G) < \frac{n}{2\log(n)}\right) \leq P\left(\alpha(G) \geq \lceil 2\log(n) \rceil\right)$$

Let X denote the number of distinct independent sets of G of size $a := \lceil 2\log(n) \rceil$.

Recall,

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1} = \frac{n!}{k!(n-k)!}$$

in particular, $\binom{n}{k} \leq n^k/k!$. Using these facts, among others, we show that the expected value of X is less than $\frac{1}{n}$.

$$\begin{aligned} E(X) &= \binom{n}{a}\left(\frac{1}{2}\right)^{\binom{a}{2}} \\ &< \frac{n^a}{a!}\left(\frac{1}{2}\right)^{\frac{a(a-1)}{2}} \\ &= \frac{n^a}{a!}\left(\left(\frac{1}{2}\right)^{\frac{a}{2}}\right)^{a-1} \\ &= \frac{n}{a!}\left(\frac{n}{2^{\frac{a}{2}}}\right)^{a-1} \end{aligned} \tag{3.1}$$

Before we proceed with the derivation of the desired upper bound on the expected value of X, we need the fact that $\frac{n}{2^{\frac{a}{2}}}$ is at most 1, since

$$2^{\frac{a}{2}} = 2^{\frac{\lceil 2\log(n)\rceil}{2}} \geq 2^{\frac{2\log(n)}{2}} = 2^{\log(n)} = n$$

Using this and the fact that $a \geq 2$, we obtain

$$E(X) \leq \frac{n}{a!} \tag{3.2}$$

Thus, in order to show $E(X) < \frac{1}{n}$ is suffices to show that $n^2 < a!$. This is indeed the case as the following elementary calculus argument shows.

Claim 3.1.3. *For $n \geq 9$, we have $n^2 < a!$.*

Proof of Claim 3.1.3. Using induction, it is easy to show that $\left(\frac{k}{3}\right)^k < k!$ for any positive integer k. Thus, in order to show $n^2 < a!$, it suffices to show $n^2 \leq \left(\frac{a}{3}\right)^a$ – so let's do just that.

$$\begin{aligned} & n^2 \leq \left(\tfrac{a}{3}\right)^a \\ \Longleftrightarrow \quad & n^2 \leq \left(\frac{\lceil 2\log(n)\rceil}{3}\right)^{\lceil 2\log(n)\rceil} \\ \Longleftarrow \quad & n^2 \leq \left(\frac{2\log(n)}{3}\right)^{2\log(n)} \\ \Longleftrightarrow \quad & n^2 \leq \left(\frac{2\log(n)}{3}\right)^{\log(n^2)} \\ \Longleftrightarrow \quad & \log(n^2) \leq \log\left(\left(\frac{2\log(n)}{3}\right)^{\log(n^2)}\right) \\ \Longleftrightarrow \quad & \log(n^2) \leq \log(n^2)\log\left(\frac{2\log(n)}{3}\right) \\ \Longleftrightarrow \quad & 1 \leq \log\left(\frac{2\log(n)}{3}\right) =: f(n) \end{aligned}$$

Since the function f is strictly increasing and $n \geq 9$, we obtain $1 < f(n)$. Thus, the desired result holds for $n \geq 9$. □

Thus, $E(X) < \frac{1}{n}$. Next, we need to show that with high probability G has no K_ℓ-subdivision, where $\ell = \lceil 8\sqrt{n} \rceil$.

We say that a subset U of $V(G)$ is a $\frac{3}{4}$-clique if the subgraph of G induced by U has at least $\frac{3}{4} \cdot \binom{|U|}{2}$ edges.

Observation 3.1.4. *For any r, the centers of every K_r-subdivision of G on at most $\frac{r^2}{8}$ vertices must induce a $\frac{3}{4}$-clique of size r.*

Proof of Observation 3.1.4. Suppose that G has a K_r-subdivision H on at most $\frac{r^2}{8}$ vertices. Then, H contains at most $\frac{r^2}{8} - r$ non-center vertices. In the K_r-subdivision H we have some centers directly connected by edges, and other centers connected by internally vertex-disjoint paths. How many such internally vertex-disjoint paths can H contain? Well, at most the number of non-center vertices. In total the K_r-subdivision must contain exactly $\binom{r}{2}$ connections (edges and/or paths) joining the centers. Thus, H contains at least $\binom{r}{2} - (\frac{r^2}{8} - r)$ edges joining centers of H. In order to show that the centers of H induce a $\frac{3}{4}$-clique we only need to show that

$$\binom{r}{2} - \left(\frac{r^2}{8} - r\right) \geq \frac{3}{4}\binom{r}{2}$$

which is elementary, and we omit the details. □

Suppose G had a K_ℓ-subdivision H. Then $n(H) \leq n(G) < \frac{(8\sqrt{n})^2}{8} \leq \frac{\ell^2}{8}$, and so, according to Observation 3.1.4, the centers of H induces a $\frac{3}{4}$-clique of size ℓ.

Thus, it suffices to show that, with high probability, G has no $\frac{3}{4}$-clique of size ℓ.

Consider any subset U of ℓ vertices and let Y denote the number of edges between these vertices.

Note that Y follows a binomial distribution $\text{bin}(\binom{\ell}{2}, \frac{1}{2})$. We shall use a special case of the Chernoff Bound (1.4), where $p = \frac{1}{2}$ and $t = \frac{1}{4}\binom{\ell}{2}$, in which case we obtain

$$\begin{aligned}
& P\left(Y - \tfrac{1}{2}\tbinom{\ell}{2} > \tfrac{1}{4}\tbinom{\ell}{2}\right) & < & \exp\left(\frac{-2\left(\frac{1}{4}\binom{\ell}{2}\right)^2}{3\binom{\ell}{2}}\right) \\
\iff \quad & P\left(Y > \tfrac{3}{4}\tbinom{\ell}{2}\right) & < & \exp\left(\frac{-2\left(\frac{1}{4}\binom{\ell}{2}\right)^2}{3\binom{\ell}{2}}\right) \\
\iff \quad & P\left(Y > \tfrac{3}{4}\tbinom{\ell}{2}\right) & < & \exp\left(\tfrac{-1}{24}\tbinom{\ell}{2}\right)
\end{aligned}$$

We just need to do some more elementary computations. Recall, $\ell = \lceil 8\sqrt{n} \rceil$, and so

$$\binom{\ell}{2} = \frac{\ell(\ell-1)}{2} \geq \frac{8\sqrt{n}(8\sqrt{n}-1)}{2} = 32n - 4\sqrt{n} > 30n$$

Thus,

$$\exp\left(\frac{-1}{24}\binom{\ell}{2}\right) = \frac{1}{\exp\left(\frac{1}{24}\binom{\ell}{2}\right)} < \frac{1}{\exp\left(\frac{30n}{24}\right)} = \exp\left(-\frac{5n}{4}\right)$$

The above computations imply

$$P\left(Y > \frac{3}{4}\binom{\ell}{2}\right) < \exp\left(-\frac{5n}{4}\right)$$

that is the probability that the ℓ-set U is a $\frac{3}{4}$-clique is less than $\exp\left(-\frac{5n}{4}\right)$. Since the U was chosen as an arbitrary subset of $V(G)$ of size ℓ, it follows that expected number of $\frac{3}{4}$-clique of size ℓ is less than

$$\binom{n}{\ell} e^{-\frac{5n}{4}} < 2^n e^{-\frac{5n}{4}} < e^{-\frac{n}{4}} \tag{3.3}$$

where we used the fact that $2^n = \sum_{k=0}^{n} \binom{n}{k} > \binom{n}{\ell}$. (This bound is, of course, very crude, but it suffices for our purposes.) Thus, by the First Moment Principle (1.2), the probability that G has a $\frac{3}{4}$-clique of size ℓ is at most $e^{-\frac{n}{4}}$.

Now, finally, we are ready to derive the desired lower bound.

$$\begin{aligned} &P\left(\left\{\chi(G) \geq \frac{n}{2\log 2}\right\} \cap \{G \text{ contains } no\ K_\ell\text{-subdiv. }\}\right) \\ &\qquad = 1 - P\left(\left\{\chi(G) < \frac{n}{2\log 2}\right\} \cup \{G \text{ contains } K_\ell\text{-subdiv. }\}\right) \end{aligned} \tag{3.4}$$

Using the subadditivity of probabilities, we find that the right-hand side of (3.4) is at least

$$1 - \left(P\left(\chi(G) < \frac{n}{2\log 2}\right) + P\left(G \text{ contains } K_\ell\text{-subdiv. }\right)\right) > 1 - \frac{1}{n} - \frac{1}{e^{\frac{n}{4}}}$$

This completes the proof of Theorem 3.1.1. □

Corollary 3.1.5. *For $n \geq 9$, the probability that a uniform random graph G on n vertices has $\alpha(G) < 2\log(n)$ is at least $1 - \frac{1}{n}$, in particular, almost every graph has independence number less than $2\log(n)$.*

Proof. From the proof of Theorem 3.1.1, we obtain $\alpha(G) \leq \lceil 2\log(n) \rceil - 1$ with high probability, and, since $\lceil 2\log(n) \rceil - 1 < 2\log(n)$, we are done. □

Corollary 3.1.5 is a special case of a more general result of Erdős [13] on independence in random graphs.

Theorem 3.1.6 (Erdős [13])**.** *A random graph in $\mathcal{G}_{n,p}$ almost surely has independence number at most $\lceil 2\ln(n)/p \rceil$.*

A proof of Theorem 3.1.6 may be found in [9, p. 336].

Corollary 3.1.7. *The probability that a uniform random graph G on n vertices has a K_ℓ-subdivision with $\ell = \lceil 8\sqrt{n} \rceil$ is less than*

$$\binom{n}{\ell} e^{-\frac{5n}{4}}$$

In particular, almost every graph G on n vertices has no K_ℓ-subdivision.

Proof. Follows from (3.3) in proof of Theorem 3.1.1. □

3.2 Girth and chromatic number

One might be inclined to think that large chromatic number would imply large clique number or, at least, small girth. However, such intuition is entirely wrong, as we shall see in this section.

Theorem 3.2.1 (Erdős [13])**.** *For each positive integer k, there exists a graph with girth at least k and chromatic number at least k.*

Proof. Consider a random graph G on n vertices, where each pair of distinct vertices $u, v \in V(G)$ is joined by an edge with probability $p \in [0, 1]$. We may assume $V(G) = [n]$. In the course of the proof we determine the appropriate value of p, but for now we consider it some fixed value in $[0, 1]$.

Set $t := \lceil 2\ln(n)/p \rceil$, and let $X(G)$ denote the number of cycles of G of length less than k. Now we intend to show that there exists a graph G' (constructed from G) with $\text{girth}(G') \geq k$ and $\chi(G') \geq k$.

We shall consider the probability that any subset S of $V(G)$ of size ℓ induces a graph containing a cycle of length ℓ. For convenience, suppose we are considering S to be the set $[\ell]$. Then each permutation of the numbers $1, 2, 3, \ldots, \ell$ corresponds, in the obvious way, to a cycle which may or may not be present in the induced graph $G[S]$. There's a catch, though! The same cycle C in $G[S]$ is represented by exactly 2ℓ different permutations of the numbers $1, 2, 3, \dots, \ell$. For example, the cycle corresponding to the permutation $1, 2, 3, \dots, \ell$ also corresponds to the following permutations[1]:

$$\begin{array}{llrllr}
(1,2,3, & \dots, & \ell-1,\ell) & (\ell,(\ell-1), & \dots, & 3,2,1) \\
(2,3,4, & \dots, & \ell-1,\ell,1) & (1,\ell,\ell-1, & \dots, & 4,3,2) \\
(3,4,5, & \dots, & \ell-1,\ell,1,2) & (2,1,\ell,\ell-1, & \dots, & 5,4,3) \\
(4,5,6, & \dots, & \ell,1,2,3) & (3,2,1,\ell, & \dots, & 6,5,4) \\
 & \vdots & & & \vdots & \\
(\ell-1,\ell, & \dots, & \ell-3,\ell-2) & (\ell-2,\ell-3, & \dots, & \ell,\ell-1) \\
(\ell,1,2 & \dots, & \ell-2,\ell-1) & (\ell-1,\ell-2, & \dots, & 2,1,\ell)
\end{array}$$

Recall, that the number of ways ℓ objects can be chosen from a total of n objects (with $\ell \leq n$) and arranged in sequences (permutations) is $P(n, \ell) = n(n-1)\cdots(n-\ell+1)$. Thus, the number of permutations of $1, 2, 3, \dots, \ell$ corresponding to *distinct* cycles of size ℓ is

$$\frac{1}{2\ell} n(n-1)\cdots(n-\ell+1)$$

The probability that a permutation of $1, 2, 3, \dots, \ell$ is realized as a cycle in G is p^ℓ. These observations enables us to derive an upper bound on the expected value of $X(G)$, the number of cycles of G of length less than k.

$$\begin{aligned}
E(X(G)) &= \sum_{\ell=3}^{k-1} \left(\frac{1}{2\ell} n(n-1)(n-2)\cdots(n-\ell+1)p^\ell \right) \\
&< \sum_{\ell=3}^{k-1} \frac{(np)^\ell}{2\ell} < \sum_{\ell=3}^{k-1} (np)^\ell < \sum_{\ell=0}^{k-1} (np)^\ell = \frac{(np)^k - 1}{np - 1}.
\end{aligned}$$

Now, Markov's Inequality implies:

$$\begin{aligned}
P(X(G) > \frac{n}{2}) &\leq P(X(G) \geq \frac{n}{2}) \leq \frac{E(X(G))}{\left(\frac{n}{2}\right)} \\
&< \frac{\left(\frac{(np)^k - 1}{np-1}\right)}{\left(\frac{n}{2}\right)} = \frac{2\left((np)^k - 1\right)}{n(np-1)}.
\end{aligned}$$

Fix p to be the value $n^{-\frac{k-1}{k}}$. (Then, certainly, $p \in [0, 1]$, as required.) With this choice of the 'edge probability' p, we obtain

$$P(X(G) > \frac{n}{2}) < \frac{2(n-1)}{n(n^{1/k} - 1)} \to 0 \quad \text{as} \quad n \to \infty \tag{3.5}$$

[1] This might be a silly illustration!

Thus,

$$P\left(\left\{X(G) \leq \frac{n}{2}\right\} \cap \{\alpha(G) \leq t\}\right) \geq 1 - P\left(X(G) \leq \frac{n}{2}\right) - P\left(\alpha(G) \leq t\right)$$
$$\geq 1 - f(n) - g(n)$$

where $f(n) \to 0$ and $g(n) \to 0$ for $n \to \infty$ as guaranteed by (3.5) and Theorem 3.1.6, respectively.

It follows that, for n sufficiently large, there exists a graph G on n vertices with independence number at most t and no more than $n/2$ cycles of length less than k. Now, finally, construct the graph G' from G by deleting one vertex of G from each cycle of length less than k. Then G' is a graph on, at least, $n/2$ vertices, and G' contains no cycle of length less than k, that is, $\mathrm{girth}(G') \geq k$. Obviously, since G' is an induced subgraph of G, the independence number of G' is at most the independence number of G, and therefore $\alpha(G') \leq t$. Altogether, we obtain

$$\chi(G') \geq \frac{n(G')}{\alpha(G')} \geq \frac{\binom{n}{2}}{t} = \frac{n}{2\lceil 2\ln(n)n^{(k-1)/k}\rceil} > \frac{n^{1/k}}{6\ln(n)} \to \infty \quad \text{as} \quad n \to \infty$$

The function $h(n) := \frac{n^{1/k}}{6\ln(n)}$ tends to infinity rather slowly as n goes to infinity. Nevertheless, h tends to infinity, and so, for sufficiently large n, we have $\chi(G') \geq k$, as desired. This completes the proof. □

The above proof of Theorem 3.2.1 is non-constructive, but there do exist constructive proofs of theorem [9, p.371]. In particular, *Mycielski's Theorem* [24], which has a nice induction proof [9, p.371], states that for any positive integer k, there exists a triangle-free k-chromatic graph.

3.3 Hadwiger's Conjecture and complete minors

Our main theorem:

Theorem 3.3.1 (Mader, 1969). *There exist an integer k such that if G is a graph with average degree $d(G) \geq 100k\ln(k)$, then G has a K_k-minor.*

To avoid trivial cases, let's just start by assume $k \geq 3$.

Corollary 3.3.2. *There exist an integer k such if G is a graph with colouring number $\mathrm{col}(G) \geq 100k\ln(k) + 1$, then G has a K_k-minor.*

Proof. If $\mathrm{col}(G) \geq 100k\ln(k) + 1$, then G has an induced subgraph H with minimum degree $\delta(H) \geq 100k\ln(k)$. Thus, $d(H) \geq 100k\ln(k)$, and so, by Theorem 3.3.1, H contains a K_k-minor, in particular, G contains a K_k-minor. □

Remark 3.3.3. *In Corollary 3.3.2, it is not so much the exact lower bound* $100k\ln(k) + 1$ *which is of interest - it is more the fact that there* exist *a function* $f \in O(k\ln(k))$ *such that, for sufficiently large* k, $\mathrm{col}(G) \geq f(k)$ *implies* $h(G) \geq k$.

A graph G is said to be *minor-balanced* if every minor of G has smaller average degree than G.

Lemma 3.3.4. *If G is minor-balanced with average degree $d(G)$, then G has a subgraph H with*

(i) order $n(H) \leq d(G)$, and

(ii) minimum degree $\delta(H) \geq d(G)/2 - 1$.

In the proof of Lemma 3.3.4 we shall be using the following observation.

Observation 3.3.5. *If G is minor-balanced with average degree $d(G)$ then for each vertex v in G and each vertex $u \in N(v)$,*

$$\frac{d(G)}{2} - 1 < |N(u) \cap N(v)|$$

Proof. Since G is minor-balanced, the minor H obtained by contracting the edge uv has average degree less than d. Hence, $2m(H)/n(H) = d(H) < d(G)$. Now, it is quite easy to see[2] that $m(H) = m(G) - 1 - |N(u) \cap N(v)|$ (just draw a little picture of the neighbourhood of $e = uv$). Altogether, we obtain

$$m(G) - 1 - |N(u) \cap N(v)| = m(H) = \frac{1}{2}d(H)n(H) < \frac{1}{2}d(G)(n(G) - 1) \quad (3.6)$$

Since $m(G) = d(G)n(G)/2$, we obtain, from (3.6),

$$\frac{1}{2}d(G)n(G) - 1 - |N(u) \cap N(v)| < \frac{1}{2}d(G)n(G) - \frac{1}{2}d(G)$$

which implies the desired lower bound on $|N(u) \cap N(v)|$. □

[2]Here we are considering vertex colouring and so all graphs are *simple* graphs, that is, graphs without multiple edges. This is important when calculating the number of edges in the minor H of G.

Proof of Lemma 3.3.4. Since G has average degree $d(G)$, it must have a vertex v of degree at most $d(G)$. Set H to be the subgraph induced by $N(v)$. Clearly, $n(H) \leq d(G)$, and, for any vertex $u \in V(H)$ $(= N(v))$, Observation 3.3.5 implies that u has at least $d(G)/2 - 1$ neighbours in $N(v)$, and so $\delta(H) \geq d(G)/2 - 1$. (In fact, $\delta(H)$ must be *strictly* greater than $d(G)/2 - 1$, but we don't need strict inequality in what follows.) □

3.3.1 Next step

Definition 3.3.6 (K_k-split-minor). *A K_k-split-minor of a graph G is a collection of disjoint subsets $V_1, \ldots, V_k \subseteq V(G)$ such that*

(a) the subgraph induced by each V_i has at most *two components, and*

(b) for each pair of distinct integers $i, j \in [k]$, there is an edge from every *component of V_i to V_j.*

We shall refer to the subsets V_i and the induced subgraphs $G[V_i]$ as parts *of the K_k-split-minor.*

Obviously, every K_k-minor is also a K_k-split-minor, but a K_k-split-minor may not be a K_k-minor. However, as shown in Lemma 3.3.11, every sufficiently large complete split-minor contains a K_k-minor.

The graph H mentioned in the following lemma may be thought of as the subgraph H found in Lemma 3.3.4.

Lemma 3.3.7. *If $n(H) \leq d$ and $\delta(H) \geq \frac{1}{2}d - 1$ for some $d \geq 100k\ln(k)$, then H has a K_{4k}-split-minor.*

The above-mentioned Lemma is an existence result - there *exists* a K_{4k}-split-minor. The existence of this split-minor will be proved using a probabilistic argument.

Molloy and Reed [23] explains that the vertex set of H is to be partitioned into $4k$ parts, $H_1, \ldots, H_{4k}$ in the following way: for each vertex $v \in V(H)$, place v into a uniformly chosen part. We need to prove the existence of a K_{4k}-split-minor of H, in particular, we need $4k$ parts. But couldn't the partitioning scheme result in some empty parts? I don't see why not, but Molloy and Reed doesn't comment on the aspect of the random partitioning of the vertex set. Just to be on the safe side, I have added, to Molloy & Reeds original proof, an argument showing that with high probability the parts $H_1, \ldots, H_{4k}$ will be non-empty.

We randomly partition the vertex set of H into $4k$ parts, $H_1, \dots, H_{4k}$, which, with positive probability, form a K_{4k}-split-minor of H.

We will require the following two properties of our partition:

Property 1 *Each vertex of H has a neighbour in every part.*

Property 2 *For each integer $i \in [4k]$, any pair $u, v \in H_i$ such that $|N(u) \cap N(v)| \geq n(H)/6$ has a common neighbour in H_i.*

Claim 3.3.8. *There exist a partitioning of the vertex set of H into $4k$ non-empty parts $H_1, \dots, H_{4k}$ we obtain a partitioning for which both Properties 1 and 2 hold.*

We postpone the proof of Claim 3.3.8, and continue with the proof of Lemma 3.3.7.

Proof of Lemma 3.3.7. According to Claim 3.3.8, there is a partition $H_1, \dots, H_{4k}$ for which Properties 1 and 2 both hold. We claim that such a partition forms a K_{4k}-split-minor of H. Property 1 implies that for each i, j there is an edge from each vertex of H_i (let alone from each component of H_i) to some vertex of H_j. Hence requirement (a) in the definition of K_k-split-minors (Definition 3.3.6) is satisfied. Now, we need only show that requirement (b) is satisfied, that is, each part H_i has at most 2 components.

Suppose that H_i has 3 vertices, x, y, z, in different components of H_i. Obviously, x, y, z are mutually non-adjacent, and so

$$|N(x) \cup N(y) \cup N(z)| \leq n(H) - 3 \tag{3.7}$$

Also, since $\delta(H) \geq \frac{1}{2}d - 1$,

$$|N(x)| + |N(y)| + |N(z)| \geq 3 \cdot (\frac{1}{2}d - 1) \geq \frac{3n(H)}{2} - 3 \tag{3.8}$$

Since

$$\begin{aligned} |N(x) \cup N(y) \cup N(z)| = |N(x)| + |N(y)| + |N(z)| - |N(x) \cap N(y)| \\ - |N(x) \cap N(z)| - |N(y) \cap N(z)|, \end{aligned} \tag{3.9}$$

it follows from (3.7) and (3.8) that

$$|N(x) \cap N(y)| + |N(x) \cap N(z)| + |N(y) \cap N(z)| \geq \frac{n(H)}{2} \tag{3.10}$$

Hence, at least one of these intersections, say $N(x) \cap N(y)$ has size at least $n(H)/6$. Now, according to Property 2, the vertices x and y has a common neighbour in H_i, which is impossible. Therefore no such three vertices x, y, z exist, and H_i has at most two components, which was requirement (b) in the definition of K_k-split-minors. $\square$

Now, we just need to prove Claim 3.3.8, which we used in the proof of Lemma 3.3.7. In the proof we use the First Moment Method.

Proof of Claim 3.3.8. Let X denote the number of vertices v which violate Property 1 and let Y denote the number of pairs u, v which violate Property 2.

Since we are considering random partitioning of the vertex set (of H), X and Y are to be considered as random variables.

For any integer i, the probability that a vertex w is assigned to the set H_i is $1/4k$, and the probability of w *not* being assigned to H_i is $(1 - \frac{1}{4k})$. Given some arbitrary vertex v_j of H, let X_j denote the indicator variable indicating whether v_j has a neighbour in every part of H. The probability that no neighbour of v_j is placed into H_i is

$$(1 - \frac{1}{4k})^{\deg(v)}$$

Thus, the probability $P(X_j)$ that there exist some integer $i \in [4k]$ for which the vertex v_j has no neighbour in H_i, is, by the subadditivity of probabilities, at most

$$4k \cdot (1 - \frac{1}{4k})^{\deg(v)}$$

which is less than

$$4k \cdot (1 - \frac{1}{4k})^{\frac{1}{2}d-1}$$

Now, $X = \sum_{j=1}^{n(H)}$, and so we obtain

$$\begin{aligned} E(X) &= \sum_{j=1}^{n(H)} E(X_j) \\ &= \sum_{j=1}^{n(H)} 1 \cdot P(X_j) \\ &= n(H)P(X_j) \qquad \text{(for some } j \in [n(H)]) \\ &\leq n(H)4k(1 - \frac{1}{4k})^{\frac{1}{2}d-1} \\ &< d4k \exp\left(-\frac{1}{4k}\right)^{\frac{1}{2}d-1} \\ &= 4kd \exp\left(-\frac{1}{8k}(d-2)\right), \end{aligned} \tag{3.11}$$

where we used the fact that $(1 - x) < \exp(-x)$ for every positive number x. We wish to show that the expression on the right-hand side of (3.11) is at most 1/2. In order to do so, we shall go into some tedious, elementary details, which the reader is encouraged to skip.

Observation 3.3.9. *The function* $f(x) := xe^{-(x-2)/(8k)}$ *is decreasing for* $x \geq 8k$.

Proof of Observation 3.3.9. First, we observe that

$$f(x) = xe^{-x/(8k)}e^{1/(4k)}$$

Since $e^{1/(4k)}$ is just a positive constant, we only need to show that the positive-valued function $g(x) := xe^{-x/(8k)}$ is decreasing for $x \geq 8k$, that is, we need to show that for any pair of positive numbers x_1, x_2 with $8k \leq x_1 < x_2$, we have $g(x_1) > g(x_2)$. Now, the natural logarithm is an increasing function on the positive reals and, therefore, showing $g(x_1) > g(x_2)$ is equivalent to showing $\ln(g(x_1)) > \ln(g(x_2))$. Define $h(x) := \ln(g(x))$. Now we only need to show that the function h is decreasing for $x \geq 8k$. Observe, $h(x) = \ln(x) - x/(8k)$. The function $h(x)$ is differentiable with derivative $h'(x) = 1/x - 1/(8k)$. Hence, $h'(x) < 0$ for $x > 8k$. This shows that the function h is decreasing for $x > 8k$, and therefore the function f is also decreasing for $x > 8k$. □

Since $d \geq 100k\ln(k) > 8k$, Observation 3.3.9, (3.11) and $k \geq 3$ imply

$$\begin{aligned}
E(X) &< 4k \cdot 100k\ln(k)e^{-(100k\ln(k)-2)/(8k)} \\
&< 400k^2\ln(k) \cdot e^{1/(4k)} \cdot e^{(-100/8)\cdot\ln(k)} \\
&< 400k^2\ln(k) \cdot 2 \cdot e^{\ln(k^{-25/2})} \\
&= 800k^2\ln(k) \cdot k^{-25/2} \\
&= \frac{800k^2\ln(k)}{\sqrt{k}k^{12}} \\
&< \frac{800}{\sqrt{k}k^9} < \frac{1}{20}.
\end{aligned}$$

Thus, we have obtained a sufficient upper bound on $E(X)$.

Next, we derive an upper bound on $E(Y)$. Recall, that Y denotes the number of pairs u, v which violate Property 2.

The probability that two vertices u, v of H both lie in the same part is $\frac{1}{4k}$. (Just think of the random partitioning in this way: we assign the vertices of H numbers from $[4k]$. Suppose that we first assign some number $i \in [4k]$ to u, now what is the probability that v gets assign the same number i ? By independence, that probability is obviously $\frac{1}{4k}$.) Given that two vertices u and v both lie in the same part H_i, the probability that they do *not* have a common neighbour in H_i is

$$\left(1 - \frac{1}{4k}\right)^{|N(u)\cap N(v)|}$$

Next, we need to show that the expected value $E(Y)$ is at most $\frac{1}{2}$.

Let $\mathcal{I}$ denote the set of 2-elements sets $\{u, v\}$ of $2^{V(H)}$ with $|N(u) \cap N(v)| \geq n(H)/6$. For each $\{u, v\} \in \mathcal{I}$, let $Y_{\{u,v\}}$ denote the random variable which takes the value 1 if there exists an integer $i \in [4k]$ such that $u, v \in H_i$ and u, v has no common neighbour in H_i; otherwise Y takes the value 0. Then $Y = \sum_{\{u,v\}\in\mathcal{I}} Y_{\{u,v\}}$.
Now, given some $\{u, v\} \in \mathcal{I}$, we need to calculate the probability that Y is equal to 1.

$$\begin{aligned}
&P(Y_{\{u,v\}} = 1) \\
&= P\left(\{\exists i \in [4k] : u, v \in V(H_i)\} \cap \{u, v \text{ has no common neighbour in } H_i\}\right)
\end{aligned}$$

$= P\left(\{\exists i \in [4k] : u, v \in V(H_i)\}\right)$
$\times P\left(\{u, v \text{ has no common neighbour in } H_i\} \mid \{\exists i \in [4k] : u, v \in V(H_i)\}\right)$

$= \frac{1}{4k} \times \left(1 - \frac{1}{4k}\right)^{|N(u) \cap N(v)|}.$

We're well on our way in the derivation of the upper bound on $E(Y)$, but we now we shall take a short detour to obtain an elementary, but for our purpose required fact on the exponential function.

Observation 3.3.10. *The function $f(x) = x^2 \exp(-\frac{x}{24k})$ is (strictly) decreasing for $x \geq 50k \ln(k)$.*

Proof of Observation 3.3.10. Define the function $g : \mathbb{R}_+ \to \mathbb{R}$ by

$$g(x) = \ln(f(x)) = \ln(x^2) - \frac{x}{24k}$$

Since the natural logarithm is strictly increasing on $\mathbb{R}_+$, it follows that f is strictly decreasing if and only if g is strictly decreasing (for $x \geq 50k \ln(k)$). The derivate of g with respect to x is $g'(x) = \frac{1}{x^2} 2x - \frac{1}{24k}$. Moreover,

$$g'(x) < 0 \quad \Longleftrightarrow \quad \frac{2}{x} < \frac{1}{24k} \quad \Longleftrightarrow \quad 48k < x$$

Since $k \geq 3$, we have $x \geq 50k \ln(k) > 48k$ and, therefore, $g'(x) < 0$ for all $x \geq 50k \ln(k)$, which shows g is strictly decreasing for $x \geq 50k \ln(k)$. Hence, f is strictly decreasing for $x \geq 50k \ln(k)$. $\square$

Back to the derivation of the upper bound on $E(Y)$.

$$\begin{aligned} E(Y) &= \sum_{\{u,v\} \in \mathcal{I}} E(Y_{\{u,v\}}) = \sum_{\{u,v\} \in \mathcal{I}} P(Y_{\{u,v\}} = 1) \\ &< \sum_{\{u,v\} \in \mathcal{I}} \frac{1}{4k} \left(1 - \frac{1}{4k}\right)^{|N(u) \cap N(v)|} \\ &< \binom{n(H)}{2} \frac{1}{4k} \left(1 - \frac{1}{4k}\right)^{n(H)/6} \\ &< \frac{n(H)^2}{2} \frac{1}{4k} \left(e^{-\frac{1}{4k}}\right)^{n(H)/6} \\ &= \frac{1}{8k} n(H)^2 e^{-\frac{n(H)}{24k}}. \end{aligned} \tag{3.12}$$

Now, it follows from Observation 3.3.10 that we can substitute $50k\ln(k)$ for $n(H)$ in (3.12) and preserve the inequality. Thus,

$$\begin{aligned} E(Y) &< \frac{1}{8k}(50k\ln(k))^2 e^{-\frac{50k\ln(k)}{24k}} \\ &= \frac{625}{2}k(\ln(k))^2 \frac{1}{k^{\frac{25}{12}}} < \frac{1}{2}, \end{aligned}$$

where the last inequality holds for any sufficiently large k ($k \geq 15.000$ is sufficient).

Now, what remains is the question of whether any of the parts $H_1, \ldots, H_{4k}$ are empty. Let Z_i denote the indicator variable which is 1 if H_i is empty and 0 otherwise, and define the random variable Z to be $\sum_{i=1}^{n} Z_i$. Then Z denotes the number of empty parts.

The probability that a given vertex v_i is not in part H_j is $(1 - \frac{1}{4k})$. Hence the probability that H_j is empty is, since the vertices are assigned to parts independently, equal to $(1 - \frac{1}{4k})^{n(H)}$, that is,

$$P(Z_j = 1) = \left(1 - \frac{1}{4k}\right)^{n(H)}$$

Now, using $n(H) \geq \frac{d}{2} - 1 \geq 50k\ln(k) - 1 > 49k\ln(k)$ and the fact that $(1 - x) < e^{-x}$ for all positive numbers x, we able to derive the desired upper bound on the expected value of Z.

$$\begin{aligned} E(Z) &= \sum_{i=1}^{4k} E(Z_i) = \sum_{i=1}^{4k} P(Z_i = 1) = \sum_{i=1}^{4k} \left(1 - \frac{1}{4k}\right)^{n(H)} \\ &= 4k\left(1 - \frac{1}{4k}\right)^{n(H)} < 4k\left(1 - \frac{1}{4k}\right)^{49k\ln(k)} \\ &< 4k\left(e^{-\frac{1}{4k}}\right)^{49k\ln(k)} = 4ke^{-\frac{49}{4}\ln(k)} \\ &= \frac{4k}{k^{\frac{49}{4}}} < \frac{4k}{k^{12}} < \frac{1}{20}. \end{aligned}$$

Therefore, by the First Moment Principle (1.2),

$$P(X > 0) < \frac{1}{20}, \qquad P(Y > 0) < \frac{1}{2}, \qquad \text{and} \qquad P(Z > 0) < \frac{1}{20}$$

which, by the subadditivity of probabilities, implies

$$P(\{X > 0\} \cup \{Y > 0\} \cup \{Z > 0\}) \leq P(X > 0) + P(Y > 0) + P(Z > 0) < 1$$

Hence, the complementary event $\{X = 0\} \cap \{Y = 0\} \cap \{Z = 0\}$ must have strictly positive probability, in particular, there exist a partition of $V(H)$ such that both $X = Y = Z = 0$, that is, a partition of $V(H)$ into $4k$ non-empty parts satisfying Properties 1 and 2. This, finally, completes the proof of Claim 3.3.8. □

3.3.2 Next step

Lemma 3.3.11. *Every K_{4k}-split-minor has a K_k-minor.*

Note the there is nothing probabilistic about the proof of Lemma 3.3.11.

Proof of Lemma 3.3.11. We start out be making some simplifying assumptions on the structure of the K_k-split-minor, $H_1, \ldots, H_{4k}$. (Here $H_1, \ldots, H_{4k}$ denotes the *parts* of the K_k-split-minor.)

By contracting each component of each part H_i into a vertex, we will assume that each part H_i consists of 1 vertex or 2 (non-adjacent) vertices.

In fact, we may assume that each part H_i has exactly 2 vertices. To see this, suppose ℓ parts, say $H_1, \ldots, H_\ell$, each have one vertex. Now, $4k - \ell > 4(k - \ell)$, so if we could only show that the remaining $K_{4k-\ell}$-split-minor with parts $H_{\ell+1}, \ldots, H_{4k}$ contains a $K_{k-\ell}$-minor $\{V_{\ell+1}, \ldots, V_k\}$, then $\{H_1, \ldots, H_\ell, V_{\ell+1}, \ldots, V_k\}$ is a K_k-minor. (The parts H_i and H_j for $i, j \in [\ell]$ with $i \neq j$ are,by definition of complete split-minors, adjacent. Suppose w is a vertex of some set V_j. Then $w \in H_m$ $(m > \ell)$ for some part H_m containing two vertices w and u. Again, by definition, each vertex $H_1, \ldots, H_\ell$ is adjacent to each vertex of H_m, in particular, each vertex $H_1, \ldots, H_\ell$ is adjacent to w, a vertex of V_j. This shows that $\{H_1, \ldots, H_\ell, V_{\ell+1}, \ldots, V_k\}$ is a K_k-minor.)

Thus, in order to prove Lemma 3.3.11, it suffices to prove the following.

Observation 3.3.12. *Suppose that the vertices of a graph S are paired $\{a_1, b_1\}, \ldots, \{a_{4k}, b_{4k}\}$ such that a_i and b_i are nonadjacent for every $i \in [4k]$, and suppose that every vertex has (at least) one neighbour in every pair but its own. Then S has a K_k-minor.*

Molloy and Reed gives the following intuition on why Observation 3.3.12 hold: If the pairs can be labeled in such a way that most of the a_i's are adjacent to each other then S has a large clique. Otherwise, the edges must "cross" in such a way that we may combine three pairs into a 'triples' (set of six vertices) many of which induces connected subgraphs. Since each triple also contains a pair, and pairs are mutually connected by edges, it follows that we obtain a large clique-minor.

We say that a triple of pairs $H_{i_1}, H_{i_2}, H_{i_3}$ is a *connected triple* if their union induces a connected subgraph.

Proof of Observation 3.3.12. The subgraph induced by any triple of pairs has 6 vertices and minimum degree at least 2. It is also easy to see that every triple of pairs either induces two disjoint triangles, or is a connected triple (okay - follows from definition of the graph S).

Any collection of ℓ pairs which does not contain a connected triple, must induce two disjoint ℓ-cliques.

By considering a maximum-sized collection $\mathcal{C}$ of disjoint connected triples, one can easily show that S has either (1) a k-clique or (2) a collection of k disjoint connected triples. Let's just think about this for a minute. If $|\mathcal{C}| \geq k$, then (2) holds. Suppose $|\mathcal{C}| < k$. Let R denote the vertices of S not contained in any of the triples of $\mathcal{C}$. It follows from the definition of S, that R consists of precisely $|R|/2$ pairs. The maximality of the collection $\mathcal{C}$ of disjoint connected triples, implies that no triple of pairs in R form a *connected* triple. Hence, the vertices of R induce two disjoint $|R|/2$-cliques. Since

$$\frac{|R|}{2} = \frac{8k - 6|\mathcal{C}|}{2} = 4k - 3|\mathcal{C}| > 4k - 3k$$

it follows that (1) hold.

If (1) holds, then we are happy. On the other hand, if (2) holds, then the connected triples of $\mathcal{C}$ form a K_k-minor of S. □

This, finally, completes the proof of Lemma 3.3.11. □

3.4 The Lovász Local Lemma

When applying the Local Lemma it is often convenient to consider dependency (di)graphs. Suppose we are given some set $\mathcal{E} = \{A_1, \ldots, A_n\}$ of events and sets (of indices) $N_1, \ldots, N_n \subseteq [n]$ such that each event A_i is independent of the set $\{A_j \mid j \notin N_i\}$ of events. Observe, that by our definition of independency, $i \in N_i$. Now, a *dependency digraph (of $\mathcal{E}$)* is a digraph D with vertex set $V(D) = \{v_1, \ldots, v_n\}$, where the vertex v_i corresponds to the event A_i ($i \in [n]$), and arc set

$$A(D) = \big\{(v_i, v_j) \mid v_i, v_j \in V(D), v_i \neq v_j, j \in N_i\big\}$$

where arcs corresponds to dependencies, that is, if a vertex v_i of D has no arc leading to any vertex of $\{v_j, \ldots, v_k\} \not\ni v_i$, then the event A_i is independent of the set $\{A_j, \ldots, A_k\}$ of events.

Theorem 3.4.1 (Lovász Local Lemma [14])**.** *Suppose that $\mathcal{E} = \{A_1, \ldots, A_n\}$ is a set of (typically bad) events, and suppose that D is a dependency digraph of $\mathcal{E}$ with maximum out-degree d. If, for all $i \in [n]$,*

$$P(A_i) \leq \frac{1}{4d} \tag{3.13}$$

Then

$$P(\overline{A_1} \ldots \overline{A_n}) > 0 \tag{3.14}$$

For convenience, we refer to Theorem 3.4.1 as the *Local Lemma*. The original proof of the Local Lemma given by Erdős and Lovász [14] is both short and elegant. I've tried to replicate their proof, and at the same time reduce the work left to the reader. A proof similar to the original proof may be found in [30].

Proof. (Erdős and Lovász [14]) Firstly, we see that it suffices to establish the following claim.

Claim 3.4.2. *For any event $A_i \in \mathcal{E}$ and any collection of events $\mathcal{F}(A_i)$ of $\mathcal{E} \backslash \{A_i\}$, define*

$$F := \bigcap_{A \in \mathcal{F}(A_i)} \overline{A}$$

Then

$$P(A_i \mid F) \leq \frac{1}{2d} \tag{3.15}$$

Suppose that Claim 3.4.2 holds. We may assume, by induction and (3.13), that, for any $j \in [n-1]$ and indices $i_1, \ldots, i_j \in [n]$,

$$P(\overline{A_{i_1}} \ldots \overline{A_{i_j}}) > 0$$

Then, by definition of the conditional probability,

$$P(\overline{A_i} | \overline{A_{i+1}} \ldots \overline{A_n}) = \frac{P(\overline{A_i A_{i+1}} \ldots \overline{A_n})}{P(\overline{A_{i+1}} \ldots \overline{A_n})}$$

for any $i \in [n-1]$, and so we obtain

$$P(\overline{A_n}) \cdot \prod_{i=1}^{n-1} P(\overline{A_i} | \overline{A_{i+1}} \ldots \overline{A_n}) = P(\overline{A_1 A_2} \ldots \overline{A_n}) \tag{3.16}$$

by simply unfolding the product and cancelling identical terms in numerators and denominators. Thus, using Claim 3.4.2, we obtain

$$\begin{aligned} P(\overline{A_1 A_2} \ldots \overline{A_n}) &= P(\overline{A_n}) \prod_{i=1}^{n-1} P(\overline{A_i} | \overline{A_{i+1}} \ldots \overline{A_n}) \\ &= (1 - P(A_n)) \prod_{i=1}^{n-1} \left[1 - P(A_i | \overline{A_{i+1}} \ldots \overline{A_n})\right] \\ &\geq \left(1 - \frac{1}{2d}\right) \prod_{i=1}^{n-1} \left[1 - \frac{1}{2d}\right] > 0 \end{aligned}$$

which proves (3.14). Thus, we need only prove that Claim 3.4.2 holds.

Argument for Claim 3.4.2. We'll prove the claim by induction on $|\mathcal{F}(A_i)|$.

Induction base: For any $A_i \in \mathcal{E}$, if $\mathcal{F}(A_i) = \emptyset$, then $P(A_i \mid F) = P(A_i) \leq 1/(4d) < 1/(2d)$. Let's do one more! If $|\mathcal{F}(A_i)| = 1$, say $\mathcal{F}(A_i) = \{A_j\}$, then $P(\overline{A_j}) = 1 - P(A_j) \geq 1 - 1/(4d) > 1/2$ and

$$P(A_i \mid \overline{A_j}) = \frac{P(A_i \cap \overline{A_j})}{P(\overline{A_j})} \leq \frac{P(A_i)}{P(\overline{A_j})} \leq \frac{1}{4d} \Big/ \frac{1}{2} = \frac{1}{2d}$$

This settles the base case.

Induction hypothesis: For some $k \geq 1$, any $A_i \in \mathcal{E}$ and any $\mathcal{F}(A_i) \subseteq \mathcal{E} \backslash \{A_i\}$ with $|\mathcal{F}(A_i)| \leq k$,

$$P(A_i \mid F) \leq \frac{1}{2d} \tag{3.17}$$

where $F = \cap_{A \in \mathcal{F}(A_i)} \overline{A}$.

Induction step: Let's consider an arbitrary event in $\mathcal{E}$, say w.l.o.g. A_1, and a subset $\mathcal{F}(A_1)$ of $\mathcal{E} \backslash \{A_1\}$ with $|\mathcal{F}(A_1)| = k + 1$.

If A_1 is independent of the set of events $\mathcal{F}(A_1)$, then (3.15) follows immediately. So suppose this isn't the case. This implies that the vertex v_1 of the dependency digraph D has at least one out-neighbour among the vertices of D corresponding to events of $\mathcal{F}(A_1)$. We may suppose that the out-neighbours of v_1 in D, and their corresponding events, have been labelled $v_2, \ldots, v_q$ and $A_2, \ldots, A_q$, respectively. We have, at least, $q \geq 2$. The maximum out-degree of D implies $q - 1 \leq d$.

Case 1. Suppose $\{A_2, \ldots, A_q\} = \mathcal{F}(A_1)$. Then $|\mathcal{F}(A_1)| = q - 1 \leq d$. If $P(\overline{A_2} \ldots \overline{A_q}) = 0$, then

$$P(A_1|\overline{A_2} \ldots \overline{A_q}) = P(A_1) \leq \frac{1}{4d} < \frac{1}{2d}$$

Hence, we may assume $P(\overline{A_2} \ldots \overline{A_q}) > 0$. Now, the desired result is established by the following inequalities.

$$\begin{aligned} P(A_1|\overline{A_2} \ldots \overline{A_q}) &= \frac{P(A_1\overline{A_2} \ldots \overline{A_q})}{P(\overline{A_2} \ldots \overline{A_q})} \\ &\leq \frac{P(A_1)}{P(\overline{A_2} \ldots \overline{A_q})} < \frac{P(A_1)}{(3/4)} \leq \frac{1}{4d}\Big/\frac{3}{4} = \frac{1}{3d} < \frac{1}{2d} \end{aligned}$$

where we used the fact that

$$\begin{aligned} P(\overline{A_2} \ldots \overline{A_q}) = 1 - P(A_2 + \cdots + A_q) &\geq 1 - \sum_{i=2}^{q} P(A_i) \\ &\geq 1 - (q-1)\frac{1}{4d} \geq 1 - d\frac{1}{4d} = \frac{3}{4} \end{aligned}$$

Case 2. Now, suppose $\mathcal{F}(A_1)$ contains more than just the events $A_2, \ldots, A_q$, say $\mathcal{F}(A_1) = \{A_2, \ldots, A_q, A_{q+1}, \ldots, A_{k+2}\}$ (where $q + 1 \leq k + 2$). Just as above, we may assume $P(\overline{A_2} \ldots \overline{A_{k+2}}) > 0$. By the definition of conditional probability,

$$P(A_1 \mid \overline{A_2} \ldots \overline{A_{k+2}}) = \frac{P(A_1 \ldots \overline{A_q} \mid \overline{A_{q+1}} \ldots \overline{A_{k+2}})}{P(\overline{A_2} \ldots \overline{A_q} \mid \overline{A_{q+1}} \ldots \overline{A_{k+2}})} \tag{3.18}$$

Since A_1 is independent of the set $\{A_{q+1}, \ldots, A_{k+2}\}$ of events, it follows from Proposition 1.1.4 that $P(A_1 \mid \overline{A_{q+1}} \ldots \overline{A_{k+2}}) = P(A_1)$, and so we obtain

$$P(A_1\overline{A_2} \ldots \overline{A_q} \mid \overline{A_{q+1}} \ldots \overline{A_{k+2}}) \leq P(A_1 \mid \overline{A_{q+1}} \ldots \overline{A_{k+2}}) = P(A_1) \leq \frac{1}{4d} \tag{3.19}$$

For any integer $i \in \{2, \ldots, q\}$, the set $\{A_{q+1}, \ldots, A_{k+2}\}$ is a subset $\mathcal{F}\backslash\{A_i\}$ of size $\leq k$, and so the induction hypothesis implies

$$P(A_i \mid \overline{A_{q+1}} \ldots \overline{A_{k+2}}) \leq \frac{1}{2d}$$

We use this, and the fact that $q - 1 \leq d$, to derive the following inequality.

$$\begin{aligned} P(\overline{A_2} \ldots \overline{A_q} \mid \overline{A_{q+1}} \ldots \overline{A_{k+2}}) &= 1 - P(\overline{A_2} + \cdots + \overline{A_q} \mid \overline{A_{q+1}} \ldots \overline{A_{k+2}}) \\ &\geq 1 - \sum_{i=2}^{q} P(A_i \mid \overline{A_{q+1}} \ldots \overline{A_{k+2}}) \\ &= 1 - (q-1)\frac{1}{2d} \geq \frac{1}{2} \end{aligned} \tag{3.20}$$

From (3.18), (3.19) and (3.20), we obtain

$$P(A_1 \mid \overline{A_2} \dots \overline{A_{q+1}}) \geq \frac{1}{4d} \Big/ \frac{1}{2} = \frac{1}{2d}$$

This completes the proof of the claim. □

□

Another formulation of the Local Lemma may be found on page 40.

3.4.1 2-colouring of hypergraphs

We need to see some examples of how to use the Local Lemma - it seems that most textbooks on the Probabilistic Method start out with the following nice and short example.

Theorem 3.4.3 (Erdős and Lovász [14]). *If $\mathcal{H}$ is a hypergraph such that each hyperedge has size at least k and intersects at most 2^{k-3} other hyperedges, then $\mathcal{H}$ is 2-colourable.*

In fact, Erdős and Lovász proved the contrapositive of the above statement, but I prefer the above statement which gives a sufficient condition for a hypergraph to be 2-colourable.

Given some edge e of $\mathcal{H}$, we let $|e|$ denote the number of vertices of $\mathcal{H}$ covered by e.

Proof of Theorem 3.4.3. Suppose we are given a uniformly random assignment of two colours to the vertices of $\mathcal{H}$. (Such an assignment will quite possibly *not* be a proper 2-colouring of $\mathcal{H}$.) For each hyperedge e, we define A_e to be the event that e is monochromatic. The probability that A_e occurs is $2/2^{|e|} \leq 2/2^k$, since e can be monocoloured in two ways and the total number of 2-colourings of e is $2^{|e|}$.

We also define N_e to be the set of edges $f \neq e$ of $\mathcal{H}$ which intersect e. Then $|N_e| \leq 2^{k-3}$, by assumption. We intend to apply the Local Lemma to the set of events to the set of events $\mathcal{E} = \{A_e \mid e \in E(\mathcal{H})\}$. (That's clearly the right thing to do, right?) In order to do so we need to establish Claim 3.4.4, which implies that $\mathcal{E}$ has a dependency digraph with maximum out-degree d at most 2^{k-3}.

Claim 3.4.4. *Each event A_e is independent of the set $\{A_f \mid f \notin N_e \cup \{A_e\}\}$.*

Let's just, for now, suppose that Claim 3.4.4 holds. If $d = 0$, then no two edges of $\mathcal{H}$ intersect, and $\mathcal{H}$ is 1-colourable. Suppose $d \geq 2$. Then, in order to apply the Local Lemma, we just need to check that $P(A_e) \leq 1/(4d)$. But this follows immediately, since $P(A_e) \leq 2^{1-k}$ and $d \leq 2^{k-3}$.

In the remainder of the proof we shall use the term 2-colouring to mean a two-valued function on $V(\mathcal{H})$ (or a subset of $V(\mathcal{H})$), which possibly leaves some hyperedges monochromatic.
Argument. Suppose that the vertices of $\mathcal{H}$ are ordered $v_1, \ldots, v_n$, where $e = \{v_1, \ldots, v_t\}$. Consider distinct hyperedges $f_1, \ldots, f_r \notin N_e \cup \{e\}$, and define

$$B := A_{f_1} \cap \cdots \cap A_{f_r} \tag{3.21}$$

In order to show independence, it suffices to show $P(A_e \mid B) = P(A_e)$, and that's what we'll do. Let $\mathcal{Y}$ denote the set of the set of 2-colourings of $\mathcal{H}$ for which the event B holds.

For any 2-colouring ρ of $\mathcal{H} - V(e)$, define T_ρ to be the set of the 2^t different 2-colorings of $\mathcal{H}$ which extend ρ.

For each such colouring ρ, the set $\mathcal{Y}$ contains either all of T_ρ or none of T_ρ as a subset. Let's just make sure we see why. If $\mathcal{Y} \cap T_\rho = \emptyset$, then fine. Otherwise, there exist some 2-colouring $y \in \mathcal{Y} \cap T_\rho$. Since $y \in T_\rho$, $y = \rho$ on the vertices of $\mathcal{H} - V(e)$. Since $y \in \mathcal{Y}$, the event

$$B = A_{f_1} \cap \cdots \cap A_{f_r} \tag{3.22}$$

holds for y. Now, since $f_1, \ldots, f_r$ are all hyperedges of $\mathcal{H} - V(e)$, it follows that ρ satisfies (3.22), and so all 2-colourings $z \in T_\rho$ satisfies (3.22), which implies $z \in \mathcal{Y}$. This shows $T_\rho \subseteq \mathcal{Y}$.

It follows, from the above, that there exist ℓ 2-colourings $\rho_1, \ldots, \rho_\ell$ of $\mathcal{H} - V(e)$ such that $\mathcal{Y}$ is the disjoint union $T_{\rho_1} \cup \cdots \cup T_{\rho_\ell}$. Thus, $P(B) = \frac{2^t \ell}{2^n}$.

Within each set T_{ρ_i}, there are exactly two 2-colourings in which e is monochromatic, and so

$$P(A_e \cap B) = \frac{2\ell}{2^n}$$

Therefore, also,

$$P(A_e \mid B) = \frac{P(A_e \cap B)}{P(B)} = \frac{\left(\frac{2\ell}{2^n}\right)}{\left(\frac{2^t \ell}{2^n}\right)} = \frac{2}{2^t}$$

which, as we have already seen, is equal to the probability $P(A_e)$. This shows that A_e and B are independent, in particular, A_e is independent of the set $\{A_e \mid f \notin N_e\}$ of events.

$\diamond$

This completes the proof of Theorem 3.4.3. $\square$

3.4.2 List vertex-colouring

In this section we consider vertex colouring where the colour assigned to any vertex v must be chosen from a list L_v of colours. The colours in L_v are referred to as the *acceptable colours* for v. An *acceptable colouring* (of G) is a proper vertex-colouring of G where the colour assigned to any vertex v is chosen among the acceptable colours of v.

Obviously, if for any edge $e = uv \in E(G)$, we have $L_u \cap L_v = \emptyset$, then G has an acceptable colouring. Now the question is how much overlap between the colour-lists of adjacent vertices can we permit and still guarantee the existence of an acceptable colouring ?

Well, if the colour-lists are very long, meaning $|L_v| \geq \Delta(G) + 1$ for all $v \in V(G)$, then we can allow the overlap to be as large as possible, that is, $\min\{|L_u|, |L_v|\}$. Given this observation, let's try to consider the case we the the lengths of the colours-list are bounded from below by some arbitrary positive integer ℓ. For this case Reed [27] used a nice application of the Local Lemma to obtain the following result.

Theorem 3.4.5 (Reed [27]). *Let G denote an arbitrary graph. If*

(1) each vertex $v \in V(G)$ has at least ℓ acceptable colours, and

(2) for each vertex $v \in V(G)$ and colour $c \in L_v$, the colour c is acceptable to at most $\frac{\ell}{8}$ neighbours of v,

then G has an acceptable colouring.

Trivial observation: For the complete n-graph with all colour-lists of equal to $\{1, \ldots, \ell\}$ for some $\ell \leq n - 1$, we do not have any acceptable colouring. And the overlap between any pair of colour-lists has size ℓ. Hence, the $\frac{\ell}{8}$ cannot be replaced by ℓ.

In [27], Reed conjectured that the $\frac{\ell}{8}$ in Theorem 3.4.5 can be replaced by $\ell - 1$. (This, at least, holds for $\ell = \Delta(G) + 1$.) However, Bohman and

Holzman [8] disproved the conjecture. Haxell [19] proved that the $\frac{\ell}{8}$ can be replaced by $\frac{\ell}{2}$, while Reed and Sudakov [28] showed that for any $\ell > 0$ and sufficiently large ℓ the overlap may be as much as $\frac{1}{1+\epsilon}(\ell - 1)$. That is, asymptotically, Reed's Conjecture is true.

Proof of Theorem 3.4.5. Suppose we are given a graph G and an acceptable list of colours L_v for each vertex $v \in V(G)$ such that the conditions (1) and (2) of theorem are satisfied. If any of the lists L_v contains more than ℓ elements then we simply remove colours from the lists such that they all have size ℓ. Of course, conditions (1) and (2) remain satisfied.

For each vertex $v \in V(G)$, we randomly assign from L_v to v. For each edge $e = xy$ and colour $i \in L_x \cap L_y$, we let $A_{i,e}$ denote the event that both x and y are coloured i. We let $\mathcal{E}$ denote the set of all such events.

We shall be using the Local Lemma to show that with strictly positive probability none of the events in $\mathcal{E}$, which implies the existence of an acceptable colouring.

Obviously, for any $A_{i,e} \in \mathcal{E}$, the probability of $A_{i,e}$ occurring is simply the probability of x being assigned the colour i *and* y being assigned the colour i, that is,

$$P(A_{i,e}) = \frac{1}{\ell} \cdot \frac{1}{\ell} \tag{3.23}$$

Comparing with the Local Lemma, we let p denote the value $\frac{1}{\ell^2}$.

Next we consider the dependency between events of $\mathcal{E}$. Take some arbitrary event $A_{i,e} \in \mathcal{E}$ with, say, $e = xy$. Now $A_{i,e}$ depends only on the colours assigned to x and y. Define

$$E_x := \{A_{j,f} \mid j \in L_x \text{ and } x \in V(f)\}$$

and

$$E_y := \{A_{j,f} \mid j \in L_y \text{ and } y \in V(f)\}$$

Recall, that an event B_0 is independent of a set of events $\{B_1, \ldots, B_k\}$, if for any subset $I \subseteq [k]$ we have

$$P\left(B_0 \mid \bigcap_{i \in I} B_i\right) = P(B_0)$$

Again, $A_{i,e}$ denotes only on the colours assigned to x and y - therefore knowing whether any events of $\mathcal{E} - E_x - E_y$ occurs, does not change the probability of $A_{i,e}$ occurring. That is, the event $A_{i,e}$ is independent of the set of events $\mathcal{E} - E_x - E_y$. Comparing with the Local Lemma, we see that we have to estimate the size of the sets E_x and E_y. For any given colour $j \in L_x$, we know that j is an acceptable colour for at most $\frac{\ell}{8}$ neighbours of x, that is, for each $j \in L_x$ there is at most $\frac{\ell}{8}$ elements in E_x. Thus, $|E_x| \leq \frac{\ell}{8} \cdot |L_x| = \frac{\ell^2}{8}$ and the same upper bound holds for $|E_y|$. Now, setting $d := \frac{\ell^2}{4}$, we find that $A_{i,e}$ is independent of all but at most d events of $\mathcal{E}$. Thus, we have conditions (a) and (b) of the Local Lemma satisfied – and we only need to show that $pd \leq \frac{1}{4}$. Of course, we have $pd = \frac{1}{4}$, and so we may apply the Local Lemma to conclude existence of a colouring φ for which none of the events in $\mathcal{E}$ occur, that is, φ is an acceptable colouring. □

3.4.3 Cycles in regular digraphs

In this section we apply the Local Lemma to prove that any regular digraph contains a 'large' collection of vertex-disjoint directed cycles. The Local Lemma may be formulated as follows.

Theorem 3.4.6 (Lovász Local Lemma [14]). *Consider a set $\mathcal{E}$ of (typically bad) events such that for each $A \in \mathcal{E}$*

(a) $P(A) \leq p < 1$, and

(b) the event A is independent of a set of all but at most d of the other events.

If $4pd \leq 1$, then with strictly positive probability, none of the events in $\mathcal{E}$ occur.

Theorem 3.4.7 (Mutual Independence Principle). *Suppose $\mathcal{X} = X_1, \ldots, X_m$ is a sequence of independent random experiments.*

Suppose further that $A_1, \ldots, A_n$ is a set of events, where each A_i is determined by $F_i \subseteq \mathcal{X}$.

If $F_i \cap F_{i_j} = \emptyset$ for all indices i_j in some set $\{i_1, \ldots, i_k\}$, then

the event A_i is independent of $\{A_{i_1}, \ldots, A_{i_k}\}$.

We omit the proof of the Mutual Independence Principle, nevertheless, we shall be using it in the proof of the following result.

Theorem 3.4.8 (Alon, McDiarmid and Molloy [5], Molloy [17]). *Every k-regular directed graph G with $k \geq 2$ has a collection of $\left\lfloor \frac{k}{4\ln(k)} \right\rfloor$ vertex-disjoint directed cycles.*

Proof. (Molloy [22].) Let $c := \left\lfloor \frac{k}{4\ln(k)} \right\rfloor$. The statement of the theorem is trivially true for $k \leq 26$, since for all such values of k, we have $c \in \{0, 1\}$. Hence we may assume $k \geq 27$. We partition the vertex set $V(G)$ into c subsets, which we denote $V_1, \ldots, V_c$, and show that with strictly positive probability, each subgraph $G[V_i]$ contains a cycle. It suffices to show that with strictly positive probability each vertex $v \in V(G)$ has at least one out-neighbour in each set V_i ($i \in [c]$). (Notice that this would imply that $V_i \neq \emptyset$ and $\delta^+(G[V_i]) \geq 1$ for all $i \in [c]$, and so each subgraph $G[V_i]$ contains at least one directed cycle.)

For each vertex $v \in V(G)$, we place v in the set V_i with probability $1/c$. For each vertex $v \in V(G)$, let B_v denote the event that there is at least one part V_i that contains no out-neighbour of v. We need to avoid the events B_v for each $v \in V(G)$. We let $\mathcal{E}$ denote the set of all events B_v, where $v \in V(G)$, and apply the Local Lemma to show that with strictly positive probability none of the bad events $B_v \in \mathcal{E}$ occur. Then we have

$$\begin{aligned} P(B_v) &= P\left(\bigcup_{i=1}^{c}\{V_i \text{ contains no out-neighbour of } v\}\right) \\ &\leq \sum_{i=1}^{c} P\left(\{V_i \text{ contains no out-neighbour of } v\}\right) \\ &= \sum_{i=1}^{c}\left(1 - \frac{1}{c}\right)^k \\ &\leq c \cdot \exp\left(1 - \frac{k}{c}\right) \end{aligned}$$

By the Mutual Independence Principle, each event B_v is independent of the set S_v of events B_u with $N^+[u] \cap N^+[v] = \emptyset$. The set S_v contains all events of $\mathcal{E}$ but at most $k^2 + k + 1 \leq (k+1)^2$ events of $\mathcal{E}$. Observe that $4(k+1)^2 < 4e^{-1}\ln(k)k^2$, since $k \geq 27$, and so we have the conditions of the Local Lemma (Theorem 3.4.6) satisfied with $p = \frac{e}{4\ln(k)k^2}$ and $d = (k+1)^2$. Hence some partition of $V(G)$ such that none of the events of $\mathcal{E}$, and the desired result follows immediately. □

3.5 Degrees and choice numbers

The title of this section is taken from an important paper by Noga Alon [3] which describes the asymptotic behavior of the choice number when the minimum degree tends to infinity. In particular, this means that when the colouring number col(G) tends to infinity then the choice number ch(G) tends to infinity. Observe such an implication does *not* hold when replacing the choice number (or the list-chromatic number) with the chromatic number. For instance, for any positive integer r,

$$\chi(K_{r,r}) = 2 \qquad \text{and} \qquad \mathrm{col}(G) = r + 1$$

and so for $r \to \infty$ we have $\mathrm{col}(K_{r,r}) \to \infty$, but still $\chi(K_{r,r}) = 2$. However, when it comes to the colouring number, we have $\mathrm{ch}(G) \geq (\frac{1}{2} - o(1)) \log_2(d)$, where d denotes the colouring number of G. This follows, as we shall see, from Theorem 3.5.1. Our proof of Theorem 3.5.1 is based on a pedagogically well-structured proof by Jean-Sébastien Sereni [29]. A very nice presentation of the proof is available through the homepage of Hong-Gwa Yeh [35].

Theorem 3.5.1 (Alon [3]). *Let s denote an integer. The list chromatic number of any graph G with minimum degree*

$$\delta(G) > 2^{2s+2} \frac{(s^2+1)^2}{(\log_2 e)^2} \tag{3.24}$$

is greater than s.

Proof. The statement of the theorem is obviously true for $s = 1$. For $s = 2$, we get $\delta > 768$, and so it follows from the characterization of graphs with list chromatic number 2 that $\chi_\ell(G) > 2 = s$ [15]. Hence we may assume $s \geq 3$.

Throughout the proof, we let $n := n(G)$, $\delta := \delta(G)$ and $\mathcal{C} := \{1, 2, \ldots, s^2\}$. The set $\mathcal{C}$ will be our set of colours. We show that there exists an s-list-assignment $L : V(G) \to 2^{\mathcal{C}}$ such that $|L(v)| = s$ for all $v \in V(G)$, and G has no proper colouring c with $c(v) \in L(v)$ for all $v \in V(G)$.

The strategy is as follows. We consider a set $B \subseteq V(G)$ with lists assigned to its vertices. There are $s^{|B|}$ different colourings of the induced subgraph $G[B]$ (where each vertex is assigned a colour from its list). We would like to show that the lists of some of the remaining vertices can be chosen such that none of the colourings of B extend to a proper list-colouring of G. In other words, we seek vertices outside B such that no matter how the vertices of B are coloured using colours from their lists, the list of at least one vertex of

$V(G) \backslash B$ will be included in the set of colours assigned to its neighbours in B.

To do this, we first need to have a fair amount of vertices outside B, that is, we should control the size of B. Then, the vertices outside of B that are of interest to us should have neighbours in B whose union of lists is somehow large. This is why the set B and the lists for its vertices should be well-chosen. that is, they should fulfil certain helpful properties. By analyzing certain random choices, we are able to show that such a good choice exists. After fixing one such choice, we proceed to prove the existence of lists for some vertices outside B such that no proper list-colouring exists[3]. The details are as follows.

Each vertex of G is chosen to be a member of B independently at random with probability $\delta^{-1/2}$. Next, each vertex b of B is assigned a list $S(b)$, which is chosen independently and uniformly at random among all the s-subsets of $2^{\mathcal{C}}$. Then $|B| \sim bin(n, \delta^{-1/2})$ and $E(|B|) = n\delta^{-1/2}$. Therefore, by Markov's Inequality[4] (1.1),

$$P(|B| > 2n\delta^{-1/2}) \leq \frac{n\delta^{-1/2}}{2n\delta^{-1/2}} = \frac{1}{2} \tag{3.25}$$

A vertex $v \in V \backslash B$ is *good*[5] if for each subset T of $\mathcal{C}$ of cardinality $\lceil s^2/2 \rceil$, there is a neighbour $b \in B$ of v whose list is contained in T. Any vertex $v \in V$ which is not good, is *bad*. Certainly, every vertex of B is a bad vertex.

Claim 3.5.2. *The probability that a vertex $v \in V$ is bad is less than* $1/4$.

We will postpone the proof of Claim 3.5.2 till later, and continue the proof under the assumption that the claim is true. Let X and Y denote the sets of bad and good vertices, respectively. Then $|X|$ and $|Y|$ are random variables with $|X| + |Y| = n)$, $E(|X|) < n/4$, and by Markov's Inequality,

$$P(|X| > \frac{n}{2}) < \frac{n/4}{n/2} = \frac{1}{2} \tag{3.26}$$

[3]Don't worry if this doesn't make much sense just yet.

[4]It is possible to get a much better upper bound on $P(|B| > 2n\delta^{-1/2})$ by using the Chernoff Bound (Theorem 1.1.3, page 2).

[5]Good in the sense that such a vertex is going to hinder an extension of any colouring of the vertices of B to a colouring of all the vertices of V.

Together, (3.25) and (3.26) implies the following:

$$\begin{aligned} & P(\{|B| \leq 2n\delta^{-1/2}\} \cap \{|Y| \geq n/2\}) \\ = \;& 1 - P(\{|B| > 2n\delta^{-1/2}\} \cup \{|X| > n/2\}) \\ \geq \;& 1 - P(|B| > 2n\delta^{-1/2}) - P(|X| > n/2) > 1 - (1/2) - (1/2) = 0 \end{aligned}$$

That is, with strictly positive probability it holds that $|B| \leq 2n\delta^{-1/2}$ *and* the number of good vertices is at least $n/2$. In particular, this means that there exist some B and S such that the event $\{|B| \leq 2n\delta^{-1/2}\} \cap \{|Y| \geq n/2\}$ holds. Let's fix B and S such that $|B| \leq 2n\delta^{-1/2}$ and $|Y| \geq n/2$.

We extend the s-list-assignment S to Y by assigning a random s-subset $S(y)$ of $2^{\mathcal{C}}$ for each vertex $y \in Y$. We show that, with positive probability, there is no proper colouring of $B \cup Y$ that assigns to each vertex a colour from its list. (Okay — that's sounds just right. Notice that $B \cup Y$ may not contain all the vertices of $V(G)$, but that doesn't matter, since if $G[B \cup Y]$ isn't s-list-colourable the the supergraph G certainly isn't s-list-colourable.)

There are $s^{|B|}$ different (possibly improper) colourings of B. Let us fix an arbitrary colouring of B and estimate the probability that it can be extended to the vertices of Y (good idea!). For each $y \in Y$, let $F(y)$ denote the set of colours that appear on its neighbours belonging to B. Note that if y can be properly coloured then $S(y) \not\subseteq F(y)$.

Claim 3.5.3.

$$|F(y)| \geq \left\lceil \frac{s^2}{2} \right\rceil \tag{3.27}$$

Argument. Suppose that (3.27) doesn't hold, that is, $|F(y)| < \lceil s^2/2 \rceil$. Then $|F(y)| \leq \lfloor s^2/2 \rfloor$. Observe that

$$|[s^2] \backslash F(y)| \geq s^2 - \left\lfloor \frac{s^2}{2} \right\rfloor = \left\lceil \frac{s^2}{2} \right\rceil$$

Thus, we may let T denote a set of $\lceil s^2/2 \rceil$ colours from the set $[s^2] \backslash F(y)$. Now, no neighbour b of y in B has its colour-list entirely contained in T (since the current colour assigned to b is in $F(y)$, and $T \subseteq [s^2] \backslash F(y)$). This contradicts the fact that y is a good vertices. Hence, (3.27) holds.

◇

Notice, that since we have fixed B and the colouring of B, the set $F(y)$ is also fixed (that is, it is *not* random). The probability that y can be coloured is at most

$$\begin{aligned}
P(\{S(y) \not\subseteq F(y)\}) &= 1 - P(\{S(y) \subseteq F(y)\}) \\
&= 1 - \frac{\#\text{ outcomes with } S(y) \subseteq F(y)}{\#\text{ outcomes}} \\
&= 1 - \frac{\binom{|F(y)|}{s}}{\binom{s^2}{s}} \\
&\leq 1 - \frac{\binom{\lceil s^2/2 \rceil}{s}}{\binom{s^2}{s}} \leq 1 - 2^{-s-1}
\end{aligned}$$

where the last inequality follows from (3.29) below.

$$\begin{aligned}
\frac{\binom{\lceil s^2/2 \rceil}{s}}{\binom{s^2}{s}} &= \frac{\lceil s^2/2\rceil(\lceil s^2/2\rceil - 1)\cdots(\lceil s^2/2\rceil - s + 1)}{s^2(s^2-1)(s^2-2)\cdots(s^2-s+1)} \\
&\geq 2^{-s}\prod_{i=0}^{s-1}\frac{s^2-2i}{s^2-i} \\
&= 2^{-s}\prod_{i=0}^{s-1}\Big(1 - \frac{i}{s^2-i}\Big) \\
&\geq 2^{-s}\Big(1 - \sum_{i=0}^{s-1}\frac{i}{s^2-i}\Big) \qquad (3.28) \\
&\geq 2^{-s}\Big(1 - \sum_{i=0}^{s-1}\frac{i}{s^2-s}\Big) \\
&= 2^{-s-1} \qquad (3.29)
\end{aligned}$$

where the inequality (3.28) follows from Lemma 3.5.5 and the fact that $i/(s^2-i) \in [0,1]$ for any integer $i \in \{0, \ldots, s-1\}$.

The choices of the lists $S(y)$ for $y \in Y$ are made independently, and so the probability that a fixed colouring of B can be extended to a proper of $G[B \cup Y]$ assigning to each vertex a colour from its list is at most

$$\left(1 - 2^{-s-1}\right)^{|Y|} \leq \left(1 - 2^{-s-1}\right)^{n/2} \leq \exp\left(-n \cdot 2^{-s-2}\right)$$

since $(1-x)^z \leq \exp(-xz)$ for all positive real number x and z. Consequently, the probability that there is a proper colouring of $G[B \cup Y]$ assigning to each vertex a colour from its list is at most

$$s^{|B|} \cdot \exp\left(-n \cdot 2^{-s-2}\right) \qquad (3.30)$$

The following claim establishes an upper bound on the value given in (3.30).

Claim 3.5.4.

$$s^{|B|} \cdot \exp\left(-n \cdot 2^{-s-2}\right) \leq \exp\left(-2n \cdot \delta^{-1/2}\right) \tag{3.31}$$

Argument. Observe, that (3.24) implies

$$\frac{1}{\sqrt{\delta}} < \frac{\log_2(e)}{2^{s+1}(s^2+1)} \tag{3.32}$$

Recall, $s \geq 3$. In order to prove (3.31), it suffices to prove

$$|B| \log_e(s) - n \cdot 2^{-s-2} \leq \frac{-2n}{\sqrt{\delta}}$$

Since $|B| \leq 2n \cdot \delta^{-1/2}$ and $s \geq 3$, it suffices to prove

$$\frac{2n}{\sqrt{\delta}} \log_e(s) - n \cdot 2^{-s-2} \leq \frac{-2n}{\sqrt{\delta}}$$

which is equivalent to

$$\frac{1}{\sqrt{\delta}} \leq \frac{1}{2^{s+2}(1+\log_e(s))} \tag{3.33}$$

However, it is easy to deduce (3.33) from (3.32) and the fact that $s \geq 3$. The details are omitted.

◇

Thus, from (3.30) and (3.31) it follows that the probability that there is a proper colouring of $G[B \cup Y]$ assigning to each vertex a colour from its list is at most $\exp(-2n \cdot \delta^{-1/2})$, which is strictly less than 1. Thus, there exists at least one s-list-assignment to the vertices of Y such that $G[B \cup Y]$ cannot be properly coloured using colours from the lists, which was what we set out to prove. Now, we only need to provide an argument for the truth of Claim 3.5.2.

Argument for Claim 3.5.2. Fix a vertex $v \in V$. It belongs to B with probability $\delta^{-1/2}$. Thus,

$$\begin{aligned} P(v \text{ is bad }) &= P(v \in B \text{ and } v \text{ is bad}) + P(v \in V \backslash B \text{ and } v \text{ is bad}) \\ &\leq \delta^{-1/2} + P(v \in V \backslash B \text{ and } v \text{ is bad}) \end{aligned} \tag{3.34}$$

It is fairly straightforward to show

$$\frac{1}{\sqrt{\delta}} + \frac{1}{4} 2^{s^2} \exp\left(-\sqrt{\delta} \cdot 2^{-s-1}\right) < 1$$

Hence, by (3.34), it suffices to show

$$P(v \in V\backslash B \text{ and } v \text{ is bad}) \leq \frac{1}{4} 2^{s^2} \exp\left(-\sqrt{\delta} \cdot 2^{-s-1}\right) \tag{3.35}$$

and that is exactly what we intend to do. Suppose that $v \in V\backslash B$. Then, for each set $T \subseteq \mathcal{C}$ of cardinality $\lceil s^2/2 \rceil$, and for each neighbour u of v, it holds that

$$\begin{aligned} P(u \in B \text{ and } S(u) \subseteq T) &= P(u \in B) \cdot P(S(u) \subseteq T \,|\, u \in B) \\ &= \frac{1}{\sqrt{\delta}} \cdot \frac{\binom{\lceil s^2/2 \rceil}{s}}{\binom{s^2}{s}} \end{aligned}$$

There are $\binom{\lceil s^2/2 \rceil}{s}$ possible choices for the subset T of $\mathcal{C}$, and at least δ possible choices for the neighbour u. Consequently, the probability that there exists a set T of $\lceil s^2/2 \rceil$ colours from $\mathcal{C}$ such that each neighbour u of v is in $V\backslash B$ or has a list not contained in T is at most

$$\binom{s^2}{\lceil s^2/2 \rceil}\left(1 - \frac{1}{\sqrt{\delta}} \cdot \frac{\binom{\lceil s^2/2 \rceil}{s}}{\binom{s^2}{s}}\right)^{\deg(v)} \leq \binom{s^2}{\lceil s^2/2 \rceil}\left(1 - \frac{1}{\sqrt{\delta}} \cdot \frac{\binom{\lceil s^2/2 \rceil}{s}}{\binom{s^2}{s}}\right)^{\delta}$$

This gives us an upper bound on the probability that an arbitrary vertex $v \in V$ is in $V\backslash B$ *and* is bad.

$$\begin{aligned} P(v \in V\backslash B \text{ and } v \text{ is bad}) &= P(v \in V\backslash B) \cdot P(v \text{ is bad} \,|\, v \in V\backslash B) \\ &\leq \left(1 - \frac{1}{\delta}\right)\binom{s^2}{\lceil s^2/2 \rceil}\left(1 - \frac{1}{\sqrt{\delta}} \cdot \frac{\binom{\lceil s^2/2 \rceil}{s}}{\binom{s^2}{s}}\right)^{\delta} \end{aligned}$$

Hence, according to (3.29) and the fact that[6] $\binom{s^2}{\lceil s^2/2 \rceil} \leq 2^{s^2}4$, we deduce that

$$P(v \in V\backslash B \text{ and } v \text{ is bad}) \leq \frac{1}{4} 2^{s^2}\left(1 - \frac{2^{-s-1}}{\sqrt{\delta}}\right)^{\delta} \leq \frac{1}{4} 2^{s^2} \exp\left(-\sqrt{\delta} \cdot 2^{-s-1}\right)$$

(where we, again, used the fact that $(1-x)^z \leq \exp(-xz)$ for all positive real numbers x and z.) And so, finally, we have established (3.35), and the proof of Claim 3.5.2 is complete.

◇

□

[6]This is easily proved by induction and the fact that $s \geq 3$.

The following result, which we used in the proof of Theorem 3.5.1 is easily proved by induction.

Lemma 3.5.5. *For any t real numbers $x_1, \dots, x_t$ with $x_i \in [0,1]$ for all $i \in [t]$,*

$$\prod_{i=1}^{t}(1-x_i) + \sum_{j=1}^{t} x_j \geq 1$$

Theorem 3.5.6 (Alon [3])**.** *For any graph G,*

$$\left(\frac{1}{2} - o(1)\right)\log_2(d) \leq \mathrm{ch}(G) \leq d \tag{3.36}$$

where d denote the colouring number of G.

The following proof wasn't part of Alon's paper [3] (he probably considered it too obvious to merit a proof.) The upper bound on $\mathrm{ch}(G)$ stated in (3.36) is tight up to a constant factor[7] of $2 + o(1)$, since $\mathrm{ch}(K_{r,r}) = (1 + o(1))\log_2(r)$.

Proof. The upper bound on the choice number $\mathrm{ch}(G)$ is trivial. Thus, we need only concern ourselves with the lower bound. Let H denote an induced subgraph of G such that

$$d = \mathrm{col}(G) = \delta(H) + 1 \tag{3.37}$$

Then, by the definition of the colouring number, $\delta(G) \leq \delta(H)$. Obviously, $\mathrm{ch}(H) \leq \mathrm{ch}(G)$. We shall apply Theorem 3.5.1 to the graph H. Let s denote the largest integer such that

$$\delta(H) < 2^{2s+2}\frac{(s^2+1)^2}{(\log_2(e))^2}$$

Then

$$\delta(H) \geq 2^{2(s+1)+2}\frac{((s+1)^2+1)^2}{(\log_2(e))^2} \tag{3.38}$$

and, by Theorem 3.5.1, $\mathrm{ch}(H) > s$. Together, (3.38) and (3.37) implies

$$d - 1 \leq 2^{2(s+1)+2}\frac{((s+1)^2+1)^2}{(\log_2(e))^2} = 2^{2s+4}\frac{(s^2+2+2)^2}{(\log_2(e))^2} \tag{3.39}$$

[7] It might be more correct to say that the bound is tight up to factor $f(d)$, where $f(d) = 2 + o(1)$.

which implies

$$16 \cdot 4^s(s^2+2+2s)^2 \geq d-1$$

There exists an constant $c_1, s_0 > 0$ and such that $c_1 \cdot s^4 \geq 16(s^2+2+2s)^2$ for all $s \geq s_0$. If we assume d to be sufficiently large, then (3.39) implies that $s \geq s_0$. Thus, we have

$$c_1 \cdot 4^s \cdot s^4 \geq d-1$$

for all sufficiently large values of d. And, of course, for all sufficiently large values of s we also have $4^s \geq s^4$. Thus, $c_2 \cdot 4^s \geq d-1$ for some constant $c_2 > 0$ and all sufficiently large values of d. In fact, choosing c_2 sufficiently large, we get $c_2 \cdot 4^s \geq d$ for all sufficiently large d. By taking the logarithm, we obtain $\log_2(c_2) + 2s \geq \log_2(d)$, which implies

$$s \geq \log_2(d)\left[\frac{1}{2} - \frac{\log_2(c_2)}{2\log_2(d)}\right]$$

for all sufficiently large values of d. Since $\frac{\log_2(c_2)}{2\log_2(d)} \to 0$ for $d \to \infty$ and $\mathrm{ch}(G) \geq \mathrm{ch}(H) > s$, we have indeed proved what we set out to prove. □

3.6 Total colouring

Given some graph G and non-negative integer k, we say that a function $\varphi\colon V(G) \cup E(G) \to \mathcal{C}$ is a *total k-colouring* (of G) if $|\mathcal{C}| = k$ and any two distinct incident or adjacent elements of G are assigned different colours. More precisely,

(i) if $uv \in E(G)$, then $\varphi(u) \neq \varphi(v)$,

(ii) if $e, f \in E(G)$ are two distinct incident edges, then $\varphi(e) \neq \varphi(f)$, and

(iii) if $e = uv \in E(G)$, then $\varphi(e) \neq \varphi(u)$.

Thus, φ is both a vertex k-colouring and an edge k-colouring with the additional property that the endvertices of any edge e are assigned colours distinct from $\varphi(e)$. The total chromatic number of G, denoted $\chi_T(G)$, is, as one would expect, the minimum k for which G has a total k-colouring.

Behzad and Vizing (see [20, Sec. 4.9]) independently conjectured that every simple graph G, $\chi_T(G) \leq \Delta(G)+2$. Obviously, $\chi_T(G) \geq \Delta(G)+1$. Jensen and Toft [20, Sec. 4.9] states several results on the total chromatic number.

We begin our study of total colourings of graphs by using the Probabilistic Method to derive an upper bound on $\chi_T(G)$ in terms of $n(G)$ and $\Delta(G)$. The result was obtained independently by Häggkvist and Chetwynd [18] and McDiarmid and Reed [21].

Theorem 3.6.1. *Every graph G satisfies*

$$\chi_T(G) \leq \Delta(G) + \lceil \ln(n(G)) \rceil + 3 \tag{3.40}$$

Before we proceed with the proof, we need to define the concepts of reject edges and reject graphs. Suppose we are given a graph G, a vertex colouring φ of G and an edge colouring ε of G. Any edge $e = uv \in E(G)$ with $\varepsilon(e) \in \{\varphi(u), \varphi(v)\}$ is called a *reject edge*, and the graph induced by all the reject edges is called the *reject graph* (of G w.r.t. the vertex colouring φ and the edge colouring ε).

Proof of Theorem 3.6.1. Let G denote an arbitrary graph, n the number of vertices of G and Δ the maximum degree of G. The statement is obviously true for $n \leq 2$. Hence, assume $n \geq 3$.

Define $\ell := \lceil \ln(n) \rceil + 2$. From Brooks' Theorem[8] and Vizing's Theorem we know that G is both has a $\Delta+1$ vertex colouring and a $\Delta+1$ edge colouring. There are certainly also $\Delta+1$ vertex colourings which does use *all* $\Delta+1$ colours. (This is not the case with edge colourings, since a graph may only have Δ edges!) Let C denote such a $\Delta+1$ vertex colouring C of G using the colours $1, \ldots, (\Delta+1)$, and let $S_1, \ldots, S_{\Delta+1}$ denote the (non-empty!) colour classes (independent sets) of C.

Let D denote an arbitrary $\Delta+1$ edge colouring of G using the colours $1, \ldots, (\Delta+1)$, and let $M_1, \ldots, M_{\Delta+1}$ denote the colour classes (matchings) of D. (Some of the sets $M_1, \ldots, M_{\Delta+1}$ may be empty!)

Let $C_1, \ldots, C_{(\Delta+1)!}$ denote the $(\Delta+1)!$ vertex colourings obtained by permuting the $\Delta+1$ colours assigned to the independent sets $S_1, \ldots, S_{\Delta+1}$.

Here is what we'll do: We'll show that we can choose an integer $i \in [(\Delta+1)!]$ such that the reject graph R_i with respect to the vertex colouring

[8] Actually, we don't need the powerful statement of Brooks' Theorem, we just need the obvious fact that any finite, simple graph H, by the greedy colouring algorithm, has chromatic number at most $\Delta(H) + 1$.

C_i and the edge colouring D has

$$\Delta(R_i) \leq \ell - 1 \tag{3.41}$$

Then we just take a $\Delta(R_i) + 1$ edge colouring ε of R_i (using 'fresh' colours), and recolour the edges of G which are also in R_i according to the edge colouring ε. This produces a (proper) total colouring of G using

$$\Delta + 1 + \ell = \Delta + \lceil \ln(n) \rceil + 3$$

Before we proceed, let's just make one thing clear about the probabilistic aspect of this proof: The colourings C and D are now *fixed*, and we choose the $\Delta + 1$ vertex colouring C_i at random from the list $C_1, \ldots, C_{\Delta+1}$ of colourings which are all permutations of C.

We pick uniformly at random an integer $i \in [(\Delta + 1)!]$ and consider the colouring C_i (from the above-mentioned list) and the reject graph R_i w.r.t. C_i and D. Our goal is to show that the expected number of vertices of degree at least ℓ in R is less than one, which implies that there exist some integer $j \in [(\Delta + 1)!]$ for which the number of vertices of degree at least ℓ in the reject graph R_j is zero, that is, $\Delta(R_j) \leq \ell - 1$, as desired.

Of course, by the linearity of expectation, to show that the expected number of vertices of degree at least ℓ is less than 1, it suffices to show that for each vertex $v \in V(R_i)$,

$$P(\deg(v, R_i)) < \frac{1}{n} \tag{3.42}$$

Now we consider an arbitrary vertex $v \in V(R_i)$, and let $d := \deg(v, R_i)$ and $vu_1, \ldots, vu_d$ denote the edges incident to v in R_i. Recall, that the reason the edge vu_j is in R_i is that $D(vu_j) \in \{C_i(v), C_i(u_j)\}$. If say $D(vu_j) = C_i(v)$, then, since D is a proper edge colouring, $D(vu_k) \neq C_i(v)$ for all $k \in [d] \backslash \{j\}$. But, since $vu_1, \ldots, vu_d$ are all reject edges, it follows that we must have $D(vu_k) = C_i(u_k)$ for all $k \in [d] \backslash \{j\}$. Thus, it suffices to consider the event that there are $\ell - 1$ edges of G incident to v whose edge-colour is identical to the vertex-colour of their endpoint $\neq v$. And we need to show that the probability of this event is less than $\frac{1}{n}$.

Let E_v denote the $(\ell - 1)$-subsets of consisting of edges incident to v. Given some $A = \{vu_1, \ldots, v_{\ell-1}\}$ in E_v, let X_A denote the $D(vu_k) = C_i(u_k)$ for every $k \in [\ell - 1]$. Moreover, define the random variable X to be $\sum_{A \in E_v} X_A$.

We need to show that $P(X > 0) < \frac{1}{n}$. According to Markov's Inequality (1.2), $P(X > 0) \leq E(X)$, and so it suffices to prove $E(X) < \frac{1}{n}$. Recall, that $E(X)$ is the expected number of sets of $\ell - 1$ edges incident to v whose edge-colour conflict with the vertex-colour of their endpoint $\neq v$. By the linearity of expectation we have

$$E(X) = \sum_{A \in E_v} P(X_A) \tag{3.43}$$

Let's try to compute the probability $P(X_A)$ for some $A = \{vu_1, \ldots, vu_{\ell-1}\} \in E_v$. Define $\alpha_j := D(vu_j)$ and $\beta_j := C(u_j)$ for all $j \in [\ell - 1]$. (Make sure to note that β_j is the colour assigned to u_j by C, *not* by C_i.) We wish to compute the probability of choosing $i \in [(\Delta+1)!]$ such that the colouring C_i (a permutation of C) is obtained by replacing the colour β_j by the colour α_j for all $j \in [\ell-1]$. The colours α_j ($j \in [\ell-1]$) are all distinct, and, therefore, if the colours β_j ($j \in [\ell-1]$ are *not* all distinct, then no such colouring exist. Otherwise, if the colours β_j ($j \in [\ell-1]$) are all distinct, then we are asking for a permutation which, for each $j \in [\ell-1]$, takes β_j to α_j. This leaves us with $\Delta + 1 - (\ell - 1)$ colours which we can permute at random. As noted above, the total number of permutations is $(\Delta + 1)!$, and therefore the probability of randomly picking a permutation (or integer $i \in [(\Delta + 1)!]$ such that b_j is mapped to α_j for each $j \in [\ell - 1]$ is

$$\frac{\#\text{ 'special' permuations}}{\#\text{ permuations}} = \frac{(\Delta + 1 - (\ell - 1))!}{(\Delta + 1)!} \tag{3.44}$$

Now, there are at most $\binom{\Delta}{\ell-1}$ sets of $\ell - 1$ edges incident to v in G. Hence,

$$\begin{aligned} E(X) = \sum_{A \in E_v} P(X_A) &\leq \sum_{A \in E_v} \frac{(\Delta + 2 - \ell)!}{(\Delta + 1)!} \\ &\leq \binom{\Delta}{\ell - 1} \frac{(\Delta + 2 - \ell)!}{(\Delta + 1)!} \\ &= \frac{\Delta!}{(\Delta + 1 - \ell)! \cdot (\ell - 1)!} \cdot \frac{(\Delta + 2 - \ell)!}{(\Delta + 1)!} \\ &= \frac{\Delta + 2 - \ell}{(\Delta + 1) \cdot (\ell - 1)!} < \frac{1}{(\ell - 1)!}, \end{aligned}$$

where the last strict inequality follows from the fact that $\ell \geq 3$. Moreover, $(\ell-1)! = (\lceil \ln(n) \rceil + 1)!$ is greater than n, and so $E(X) < \frac{1}{n}$, which was what we need to prove.

Let's recapitulate the main ideas: We have shown that $E(X) < \frac{1}{n}$ and so $P(X > 0) < \frac{1}{n}$. This implies that the expected number of vertices of degree $\geq \ell$ in R_i is strictly less than 1, and, therefore, there must exist some $i \in [(\Delta+1)!]$ for which $\Delta(R_i)$ is strictly less than ℓ. Now take a $\Delta(R_i)+1$ edge colouring ε of R_i (using colours distinct from $[\Delta+1]$), recolour the edges of G according to ε to obtain a $(\Delta+1)+(\Delta(R_i)+1) \leq \Delta+\ell+1 = \Delta+\lceil \ln(n) \rceil+3$ (proper) total colouring of G. $\square$

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