Coflow, Covering Vertices by Directed Circuits, and a Lower Bound on the Stability Number of a Graph

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(joint work with Jack Edmonds)

Let $G$ be a digraph, and for each edge $e$ of $G$, let $d(e)$ be a non-negative integer. The capacity, $d(C)$, of a dicircuit $C$ means the sum of the $d(e)$’s of the edges in $C$. A version of the Coflow Theorem (1982) says:

**Coflow Theorem.** The maximum size of a subset $S$ of vertices of digraph $G$ such that each dicircuit $C$ of $G$ contains at most $d(C)$ members of $S$ equals the minimum of the sum of the capacities of any subset $H$ of dicircuits of $G$ plus the number of vertices of $G$ which are not in a dicircuit of $H$.

When we proved the Coflow Theorem, we hoped to prove the following conjecture made by Gallai in 1963:

**Gallai’s Conjecture.** For any digraph $G$ such that each edge and each vertex is in a dicircuit, the maximum number of vertices in $G$ no two of which are joined by an edge is at least as big as the minimum number of dicircuits which together cover all the vertices.

However, we were missing the following:

**Lemma.** For any digraph $G$ such that each edge and each vertex is in a dicircuit, $G$ contains a set $F$ of edges such that $G - F$ is acyclic and every edge of $G$ is in some dicircuit which contains exactly one edge of $F$.

We recently learned that Knuth proved this lemma in 1974. The Coflow Theorem together with Knuth’s Lemma provides a proof of Gallai’s Conjecture different from that recently published by Bessy and Thomassé.