Double-Critical Graph Conjecture

Definition (Double-Critical Graphs)

A vertex-critical graph G is *double-critical* if

for any edge $xy \in E(G)$: $\chi(G - x - y) \leq \chi(G) - 2$.

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Our weaker conjecture:

Conjecture

Every double-critical k-chromatic graph contains K_k as a minor.

Helpful results ...

Theorem

$m(G) \geq 2n - 2$ *implies* $G \geq K_4$.

$m(G) \geq 3n - 5$ *implies* $G \geq K_5$ *[Dirac, 1964].*

$m(G) \geq 4n - 9$ *implies* $G \geq K_6$ *[Mader, 1968].*

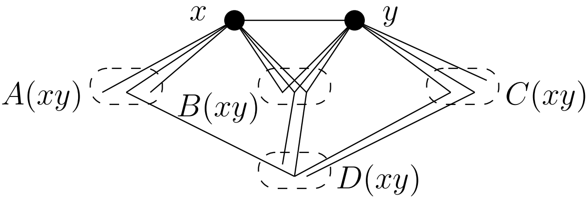
$m(G) \geq 5n - 14$ *implies* $G \geq K_7$ *[Mader, 1968].*

$m(G) \geq 6n - 19$ *implies* $G \geq K_8$ *[Jørgensen, 1994].*

$m(G) \geq 7n - 26$ *implies* $G \geq K_9$ *[Song and Thomas, 2006].*

Sketch of proof techniques

Given a double-critical k-chromatic graph $G \neq K_k$ $(k \geq 6)$.



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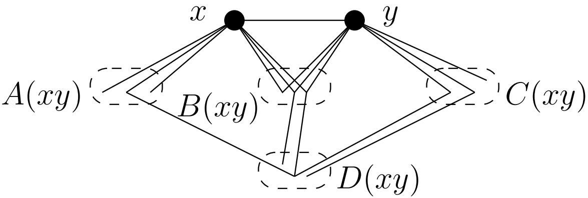

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(i) For all edges $xy \in E(G)$, $|B(xy)| \geq k - 2$.

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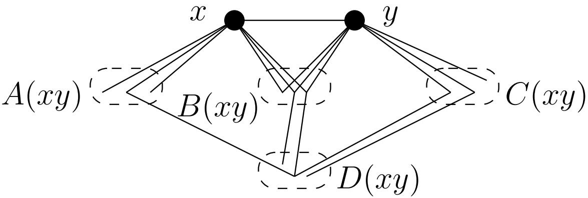

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(i) For all edges $xy \in E(G)$, $|B(xy)| \geq k-2$.

(ii) For any $x \in V(G)$, there exists $y \in N(x)$ such that $\delta(G[A(xy)]) \geq 1$.

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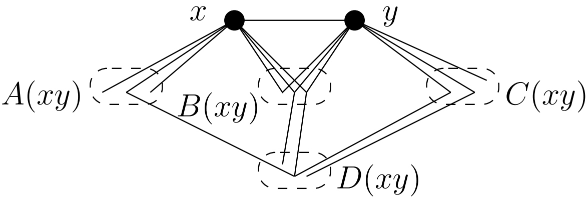

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(ii) For any $x \in V(G)$, there exists $y \in N(x)$ such that $\delta(G[A(xy)]) \geq 1$.

(iii) Each vertex of G has degree at least $k+1$.

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Theorem (Kawarabayashi, Toft, P., 2008)

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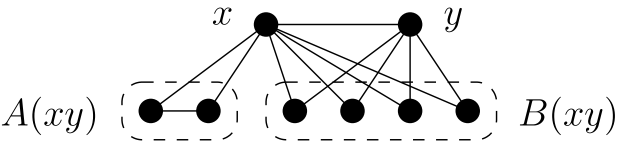

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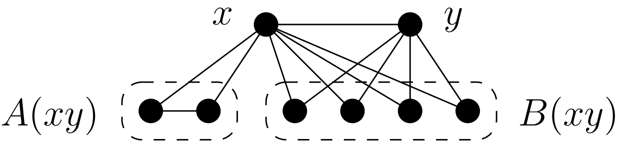

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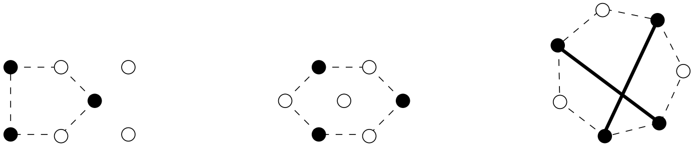

(g) $\mathrm{G[N(x)]}$ (h) $\mathrm{G[N(x)]}$ (i) $\mathrm{G[N(x)]}$

In each case we find $G \geq K_6$.

□

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If $\delta(G) = 9$, then things get ugly.

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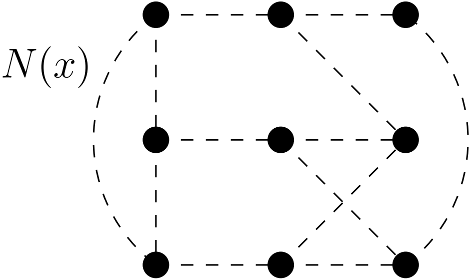

Figure: Every edge not explicitly indicated missing is present.

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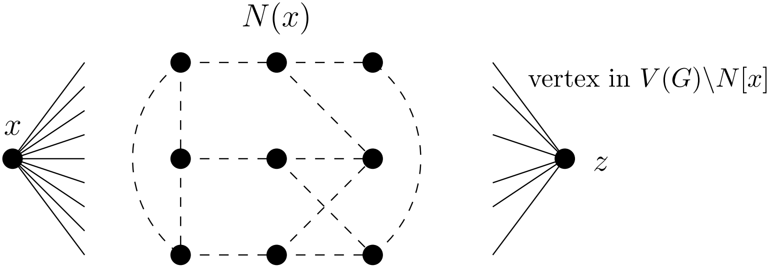

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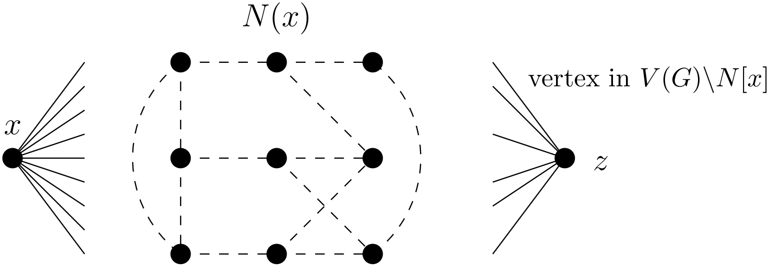

We know that G is 6-connected and $N[x] \neq V(G)$.

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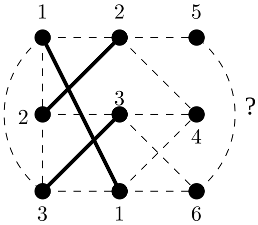

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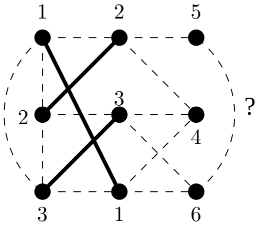

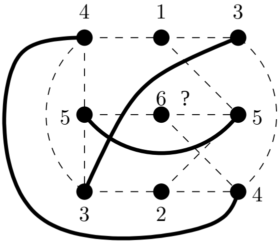

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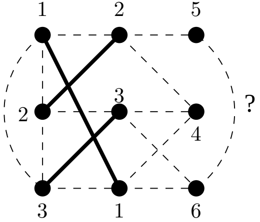

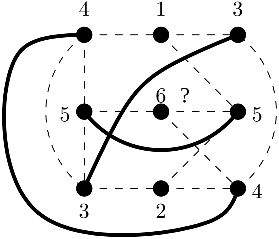

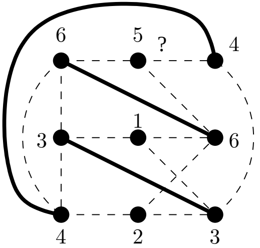

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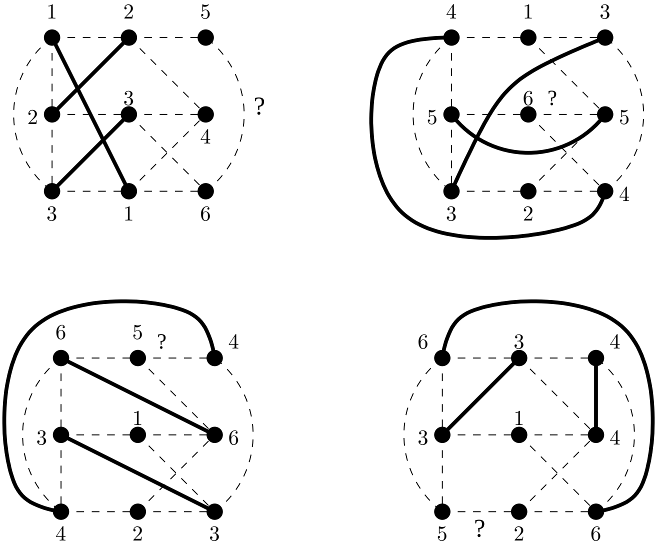

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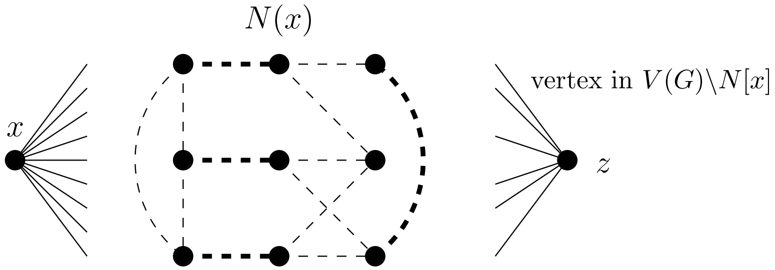

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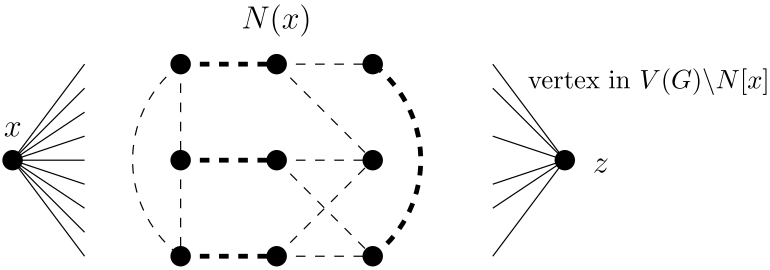

By using the 6-connectedness, $G \geq K_7$.

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(v) Settle the Double-Critical Graph Conjecture (open for $k \geq 6$).