Coloring simple uniform hypergraphs of small size

Alexandr V. Kostochka
University of Illinois at Urbana-Champaign
USA

Joint work with
M. Kumbhat and V. Rödl

Abstract

A hypergraph is *simple* if it has no 2-cycles, i.e., no two distinct edges of the hypergraph have more one vertex in common. Let $m^*(r,k)$ denote the fewest edges that might have a simple r-uniform non-k-colorable hypergraph. Erdős and Lovász proved that

$$\frac{k^{2(r-2)}}{16r(r-1)^2} \leq m^*(r,k) \leq 1600r^4k^{2(r+1)}.$$

Szabó improved the lower bound by a factor of $r^{2-\epsilon}$ for large r. We improve both, upper and lower bound for large r (in comparison to k and ϵ) to

$$k^r/r^\epsilon \leq m^*(r,k) \leq c \cdot (r \ln k)^2 k^{2r}.$$

The bounds generalize to b-simple hypergraphs, i.e. hypergraphs in which no two distinct edges share more than b vertices. We also give a new random construction of r-uniform non-k-colorable hypergraphs of arbitrary girth with maximum degree at most $\lceil r\, k^{r-1} \ln k \rceil$.