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EXPANDING THE DOMAIN - VARIABLES AND FUNCTIONS IN AN INTERDISCIPLINARY CONTEXT BETWEEN MATHEMATICS AND PHYSICS

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The lack of coordination between the curricula of physics and mathematics is in one of the primary cause of students' difficulty of application of mathematics in physics. It is difficult for the students to transfer concepts, ideas and procedures learned in mathematics to a new and unanticipated situation, e.g. in physics lessons. An alternative to this traditional transfer method stresses the importance of modelling activities in an interdisciplinary context between physics and mathematics. Physics provides mathematics with interesting problems to investigate, and mathematics provides physics with powerful tools in analyzing data and theorizing. In this paper a modelling approach to the concepts of variable and function in upper secondary school is introduced. The approach introduces the notion of functions as a tool to model relationships from physics in which two quantities vary before teaching the formal mathematics.

1. INTRODUCTION

In Denmark a structural reform is introduced in upper secondary school from August 2005. The aim of the reform is to improve young people's real study competence and strengthen the basis for getting more young people through a higher education programme. It is explicitly stated that there must be a general strengthening of the natural sciences subjects. The teaching must be modernised with a view to promoting the pupils' motivation for and interest in technology, natural sciences and scientific ways of thinking. Authentic problem complexes must be included in the teaching. The reform implies that more lessons are set-aside for optional subjects organised as subject packages and for totally free options and at the same time by allocating fewer lessons to the obligatory subjects. To prepare students for studies in natural sciences, mathematics and technology in tertiary education it is the intention that some of the subject packages should have as its core subjects mathematics and subjects of the natural sciences. An important feature of subject package is that the participating subjects form a coherent program. This will be ensured by a closer interaction between the subjects (Danish Government, 2002).

To implement the objectives of the reform cooperation across the traditional boundaries between the subjects both at the level of subject matter as well as at the level of pedagogy is required. One of the great challenges is to develop integrated learning environments for mathematics and physics. Although it's a shared view among many teachers of upper secondary school that closer relations between mathematics and physics should other things being equal help the students' to grasp both mathematics and physics, a better coordination and mutual interaction of the two subjects is far from being a trivial task. This is partly due to a lack of framework for integrating productive ideas from a variety of theoretical and practical perspectives on the relations between mathematics and physics.

2. MODELLING – THE LINK BETWEEN MATHEMATICS AND PHYSICS

A permanent theme in physics education is the adequate level of mathematics language. Mastering mathematical formalism is often a prerequisite for understanding in physics and for many newcomers in upper secondary school this formalism act as a barrier, too high to overcome. A source of the problem might be that it is often by the teachers presumed obvious that the basis mathematical facts must be apprehended before studying physics. As a consequence, it is the students' task to transfer the mathematical concepts and ideas to new setting, e.g. in physics.

2.1 Transfer and domain specificity

Transfer is one of the biggest challenges in education. It is well known that it is difficult for the students to apply concepts, ideas and procedures learned in mathematics in a new and unanticipated situation, either in or out of school. The lack of coordination between the curricula of physics and mathematics is in one of the primary causes of the students' difficulty of application of mathematics in physics. Further students get a deformed picture of how mathematics concepts are built, grow and used in physics, and so they tend to adopt a negative attitude towards mathematics and physics. The world famous Russian mathematician Vladimir Arnold puts it this way:

In the middle of the twentieth century an attempt was made to separate mathematics and physics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course, in total ignorance to any other sciences. They first began teaching their ugly scholastic pseudo-mathematics to their students, then to the schoolchildren (Arnold, 1998; p. 229).

According to Niss (1999) a significant example of the major findings of research in mathematics education is the key role of domain specificity. The student's conception of a mathematical concept is determined by the set of specific domains in which that concept has been introduced for the student. When a concept is introduced in a narrow mathematical domain, the student may see it as a formal object with arbitrary rules. This results in the recognised difficulty of application of the concept in new settings. As an alternative interdisciplinary activities between mathematics and physics offer a great variety of domain relations and context settings that can serve as a basis for developing a more practical and coherent structure of a mathematical concept. By *expanding the domain* with contexts from physics the problem of domain specificity is transcended and the curriculum is presented as a cohesive program.

2.2 Connecting mathematics and physics by modeling activities

Mathematics plays a crucial role in physics and in other fields of science and practice too. This role is brought about predominantly through the building, employment, and assessment of mathematical models. This role should also be reflected in educational settings. Mathematics has, in fact, long been a useful language for developing physics in a way, which is understandable for students and young people. Modelling by mathematization treats specifically the role of mathematics in physics, and of the link with mathematics in various fields of physics education. According to Lesh, Lester & Hjalmarson (2003) a focus on model and modelling avoids the problems of transfer and domain specifity, because models make sense of complex systems, and the purpose of the models is to provide meaningful ways for students to construct explain, describe, explain, manipulate, or predict patterns and regularities associated with complex systems. Blum & Niss (1989) emphasize the importance of close instructional contact between mathematics and physics with the aim of getting access to representative cases from physics to shed light on possibilities, conditions, difficulties and pitfalls of application and modelling.

Modelling activities emphasize the connections between mathematics and physics, which in upper secondary education are usually regarded as separate subjects. Therefore modelling activities in an interdisciplinary setting between mathematics and physics have the potential to play the key role when the domain is expanded. Further such an approach would be in concordance with the Danish upper secondary school reform's intentions of a closer interaction between the core subjects of the subject packages.

3. INTERDISCIPLINARY COMPETENCES

The reform of Danish upper secondary school calls for an innovation of the content provided on the basis of target management rather than content management. This draws attention to new ways of describing curricula. In the so-called *KOM-report* (Competences and Learning in Mathematics) the notion of competence is introduced as a basis for describing and analyzing mathematics instruction from kindergarten to tertiary level. The KOM-report defines a mathematical competence as an insightful preparedness to act properly in situations that contain a particular kind of mathematical challenge. Eight mathematically competences are identified aiming at describing what it means to master mathematics: (1) mathematical thinking competence (2) problem solving competence (3) modelling competence (4) reasoning competence (5) representational competence (6) symbolizing and formalizing competence (7) communication competence and (8) auxiliary tools competence (Niss & Jensen, 2002). In a similar report *Fremtidens Naturfaglige Uddannelser* (Science Education for the Future) four science competences are identified and described: (1) empirical competence (2) representational competence (3) modelling competence and (4) reflective competence. It is recommended that all secondary level students should be offered a specific Science Programme including the subjects of physics, chemistry and biology on an equal footing. The report doesn't deal with the role of mathematics in science education – actually mathematics is hardly mentioned in the report (Andersen et. al., 2003). The KOM-report shortly touches the relations between mathematics and science by pointning at the fact, that although a growing number of subjects include ingredients from mathematics, it still diffucult for both teachers of mathematics and teachers of other subject to see the use of mathematics in other subjects - partly due to the use of concepts and language. This paradox of relevance leads to the *problem of isolation*, which works to the disadvantage of both mathematics and other subjects, which could profit from a conscious inclusion of mathematical competences (Niss & Jensen, 2002). But although the reports only to a modest degree deal with the relations between mathematics and science two competences are present in both reports – the modelling and the representational competence. This indicate that the modelling and representational competences are the links between mathematics and science – we call them *interdisciplinary competences*.

3.1 The modelling competence

The modelling competence includes structuring an intra- or extra-mathematical situation to be modelled, mathematizing the situation, analysing and tackling the model, interpreting the results, validation of the model, communicating about the model, monitoring the modelling activity.

The conception of the modelling competence as an interdisciplinary competence, is accentuated by the attention to teaching of models and modelling in both mathematics and science education (e.g. Gilbert & Boulter, 1998; de Lange, 1996). Many mathematics and science educators are in favour of a more realistic education, where modelling activities are used to treat concepts in realistic, everyday life contexts. But little attention has been paid to the potential of such activities to bridge the gap between mathematics and physics. A cause of this might be the very simplistic way problem solving often has been conceptualized. There are two levels: (1) the realistic, everyday life contexts and applications of mathematics, and (2) the abstract world of mathematical concepts and operations. According to this perspective problem solving involves a translation from the physical reality into the abstract mathematical world. A consequence is that mathematics may be and often is learned separately from its applications. As an alternative to this approach Lester & Kehle (2003) suggest to subsume problem solving within the much broader category of mathematical activity centred on model eliciting tasks. They see a fruitful blurring of task, person, mathematic al activity, learning, applying what have been learned, and other features of mathematical problem solving. I share their view that this blurring metaphor does not express less precision. On the contrary it reflects a richer view and a more authentic view on activities in a classroom, where the students must express, test, modify, revise, and refine their own ways of thinking during the process of designing powerful conceptual tools that embody constructs that students are intended to develop. Further the artificial division in a mathematical world and reality is abolished.

Freudenthal (1991) emphasises phenomenological exploration, and argues for that the starting point for mathematics education is those phenomena that beg to be organized. Physics offers a great variety of situations that beg to be organized and structured by mathematization. For example in an interdisciplinary context between mathematics and physics the students could recognize that phenomena with periodic features are best modelled by trigonometric functions, and that radioactive decay tends to be exponential. The starting point of the Dutch approach that is known as realistic mathematics education (RME) is based on Freudenthal's (1991) view of mathematizing as the key process of mathematical activity. There are two types of mathematization in an educational context – horizontal and vertical mathematization. In horizontal mathematization, the students come up with mathematical tools, which can help to organize and solve a problem located in a real-life situation. Vertical mathematization is the process of reorganization within the mathematical system itself, like finding shortcuts and discovering connections between concepts and strategies, and the application of these discoveries. This approach espouses mathematics as an activity without loosing sight of mathematics as a product. In the RME approach the instructional designer tries to construe a set of context problems that can lead to a series of processes of horinzontal and vertical mathematization that together result in the reinvention of mathematical ideas that one is aiming for. The notion emergent model is introduced for the model emerging from the students modelling activities. The emergent models bridge the gap between informal knowledge and the formal knowledge through a transition from modelling of situations to models of as a basis for mathematical reasoning. The idea is that models, which initially refer to a concrete context that is meaningful to the students, gradually develop into general models for mathematical reasoning within a mathematical framework (Gravemeijer, 1997).

The strong emphasize on mathematization makes RME a promising theoretical framework to design instructional sequences that strengthens the relation between mathematics and physics in concordance with the gist of the reform of the Danish upper secondary school. This does not mean that that all models are or should be mathematical, but mathematization makes it possible to transfer the modelling process from one place to another. And the description of the modelling competence as an interdisciplinary competence of course entails that situations from physics are embedded in the contexts to be mathematized - a *horizontal linking* of mathematics and physics. Also the vertical mathematization must include a *vertical structuring*, that is the conceptual anchoring of the general model in the systematic and framework of mathematics and physics respectively.

3.2 The representational competence

The representational competence includes handling of representations of a diversity of matters from mathematics and physics, understanding and application of different form of representations, knowledge of strength and weakness of a representation, and selection among and translation between different forms of representations.

Almost all forms of learning involve some ways of representing information. When modelling in an interdisciplinary context the students are confronted with information from multiple sources that is presented and communicated in different forms. As the students face more complex problems in they must develop an increasingly large repertoire of representations and knowledge of how to use them productively. The point here is that the modelling and representation competencies overlap each other, and they are by no means independent in practice. Lesh & Doerr (2003) introduce the model and modelling perspective based on the assumption that models for making sense of complex systems are some of the most important components of mathematical knowledge and understanding. This approach gives the students a more accurate portrayal of what mathematicians spend most of their time doing – constructing and investigating structurally interesting systems – or using them to make sense of real-life applications. Especially it is emphasized that in the process of mathematizing the students focus on representations that have the greatest power and usefulness and go beyond thinking with models and representation systems to also think about the similarities and differences, and strengths and weakness, for a variety of purposes.

Using multiple representations allow different for a diversity of approaches, provide students contrasting insight, and rely on different sensory impressions. Confrey (1996) warns against viewing multiple representations as contributing to the ascension towards an all-encompassing form, such an algebraic representation. Instead she argues for an epistemology of multiple representations, where the difference between representations and their unique contributions are recognized and strengthened. By weaving the use of the representations together by modelling activities in an interdisciplinary context, we might avoid one of main sources of the problems transfer and domain specificity; the different use of mathematical concepts in mathematics and physics.

4. A MODELLING APPROACH TO THE CONCEPT OF FUNCTION

Interdisciplinary competences serve as a promising basis on which we can rethink the curriculum of mathematics and physics of the reformed Danish upper secondary school. It must be stressed that competences should not be regarded as a replacement of subject matter. It is combination of subject matter and competences that may lead to improved curriculum, teaching and learning. Achieving and developing competences is not merely the ability to perform certain actions or the mastery of certain skills. It is through the students' engagement in modelling activities their competences is experienced and manifested. The great challenge is therefore to develop model-based practices where the students encounter situations that provide experience in which the students can achieve and develop interdisciplinary competences. A very direct way to address this challenge is to provide a framework to implement the interaction between interdisciplinary competences and a central concept of subject matter in upper secondary school, e.g. the concept of function.

4.1 The traditional correspondence approach to the concept of function

Placed in the centre of attention of upper secondary education the concept of function has a central and organizing role around which many other important mathematical ideas resolve. Research shows that the concept of function proves to be one of the most difficult concepts to master in the learning sequence of school mathematics (e.g. Harel & Dubinsky, 1992). In mathematics education of the Danish upper secondary school the concept of function is typically introduced as correspondence, such that for each element of the domain there is exactly one element of the range. In the current curriculum links between phenomena from physics and mathematical representations of functions are introduced only after the students have been taught to think of functions as interpretations of algebraic expressions. Such an approach accentuates the island problem – the gap between the island of formal mathematics and the mainland of real human experience. Kaput (1994) elucidates the gap with the difference between mathematical functions that are defined by algebraic formulas, and empirical functions that describe every-day-life phenomena. To attack the island problem we must seek situations where the students can maximally exploit their own authentic situations to come to grips with the formal mathematics. Here it should be noted that modelling could be considered a form of mathematical challenge that supports the emergence of representations of functions (Yerushalmy & Shternberg, 2003). A sensible alternative to the traditional approach would be a modelling approach to the concept of function, grounded on the conception of a function as a tool to identify, describe and investigate phenomena. Through the recognition of changes in the surrounding world, and the identification of the relationships among these changes, the students should perceive the concept of function as a tool for making sense of reality.

4.2 Functions as a tool to identify, describe and investigate authentic phenomena

According to Sierpinska (1992) functions should first appear as an appropriate tool for mathematizing relationships between physical (or other) magnitudes. An awareness of the possible use is a sine qua non condition for making sense of the concept of function at all. Therefore the notion of function as a relationship between variables (the covariational approach) is more relevant to the students than the correspondence approach. The literature on function instruction supports the promotion of conceptual thinking about functions that includes investigations of patterns of change (e.g. NCTM, 2000). Concepts like variation, stock and flow are omnipresent in the students' daily lives including lessons in physics. Further the covariational approach provides a more profound understanding of the concept of variable. Research (e.g. White & Mitchelmore, 1996) shows that a major source of students' difficulties in applying functions is an undeveloped concept of variable. In particular, the students often treat variables as symbols to be manipulated, rather than as quantities to be related. Here it should be noticed, that when students are involved in a modelling activity they gain practice in identifying and representing variables. Representing variables in a model requires students to practice looking for structures that both extra- and intramathematical entities have in common. In an interdisciplinary activity between mathematics and physics variables represent quantities that change, and functions are the tool to study the relationships among the changing quantities.

Confrey (1996) considers what happens to the meaning of the term function when placed into a framework involving contexts and the use of multiple representations. She argues that sophisticated mathematical thought does not entail leaving the realm of everyday activity, but in forming rich connections with it. Mathematical knowledge is then described as dialectic between grounded activity and systematic enquiry. Grounded activity includes the use of contexts, multiple representations, and tools to get a feel for an idea. Movement within grounded activity is based on example, experience and knowledge of one's resources. Seeing functions as tools anchors them as a form of grounded theory. Systematic inquiry, in contrast, describes ways of codifying the grounded activity, of creating languages and symbol systems that allow one to move about logically and analytically, without reference back into the system of grounded activity. By emphasizing that the combination of multiple representations with context is the basis for a modelling approach to functions, the dialectic between the grounded activity and systematic inquiry can serve as a useful theoretical framework for a combination of subject matter and interdisciplinary competences that may lead to improved learning of the concept of functions.

5. INTERDISCIPLINARY COURSES

If reform is the aim, prototypes of instructional sequences with learning materials that are in harmony with new perspectives must be available to the teachers. At University of Southern Denmark we are a team of educational researchers and teachers from upper secondary school, who try to find out how the idea of a modelling approach to function in interdisciplinary context can be implemented. With interdisciplinary competences as a frame we develop, implement, and evaluate courses with integrated tasks for mathematics and physics. The work is based on the model of educational reconstruction that closely links analytical and empirical educational research with development of teaching and learning sequences. The model involves three main components which mutually interact: First, analysis of the content structure (including the educational viewpoint); second, the execution of empirical investigations which at first have explorative character; and third the construction of instructional units. These three components are supposed to stimulate each other in an interactive and cyclic process (Katmann et. al. 1997). A detailed planning of an instructional sequence is developed as a didactical structure, designed in the form of a scenario of what is expected to happen in an instructional sequence. An important aspect of designing a didactical structure is that of comparing the devised didactical structure, as a prediction of what was expected to happen, to what an interpretation of what actually does happen when it is tried. In this way, the didactical structure goes empirical and thus become open for revisions.

5.1 Radioactivity and exponential growth in an interdisciplinary context

As a result of our work there exist examples of integrated mathematic-physics courses that encourage the notion that mathematical functions can be used the model relationships in which two quantities vary. The first course was designed and implemented in 2000 as a six weeks integrated mathematics-physics course for a group of 26 grade 10 students. The course was centred on exponential growth and radioactivity. The exponential behaviour of the accumulation of the variable number of nuclei is a very common phenomenon in our physical, biological and social environments and therefore the same model, structure and behaviour can be found in the study of other parts of the real world. The overall curricular unit was designed to integrate three components; the gathering of data from a physical experiment, development and exploration of mathematical models (verbal, symbolic, graphical and tabular) and a reflective discourse about the mathematical content of the models. To make the structure and content of mathematics and physics as explicit as possible, course material was organised around a small number of basic models embedded in context problems. Instruction was designed to make the students familiar with the structure and use of the basic models. Beginning with an experimental situation, the students had to make explicit their own representations of a physical event, choose variables and pose relationships among them. In this manner, e.g. acting with the tabular representation of data, the tabular changes from functioning as a model of acting with the collected data, to functioning as a model for reasoning about the relations between two variables. By this approach the students developed a concept of function, which included both the traditional and static correspondence notion, that treats functions as a mapping from x to a corresponding y, and the more dynamical covariational notion, that relates the change in x to the change in y. This facilitated the students' functional thinking and demonstrated how by concentrating on the concept of functions, students can lay a foundation for two essential mathematical understandings: the ability to model authentic phenomena mathematically and the capacity to create graphic, linguistic, and numerical representations of mathematical information.

It was the basic idea of the integrated course to use experimental contexts from physics, to expand the domain for the acquisition of the concept of exponential functions. This expansion offered the students an environment, where a reasoned analysis of authentic situations required moving between the different representations of the quantitative relations in the situations, and where passages between different settings are experienced as natural moves. An important aspect of modelling is that it allows students to visualize abstract concepts by creating structures through which they can explore and experiment. For instance the students' investigation of the simultaneous decay of two different radioactive sources naturally led to the idea of summation of two functions. The contexts from physics thus provided the students with an experience of the actions that create the need for the application of mathematizing. And the study of change offered by this approach gave the students a deeper understanding of the ways in which changes in quantities can be represented and of the concept of rate of change.

To illuminate how the students' engagement in modelling activities contributed to experience and manifest their interdisciplinary competences, we point on that the students' made reference to both mathematics and physics and used representations transcending the boundaries between the two subjects. The students' combination of perspectives from both subjects often helped them to overcome problems in the modelling process. When modelling the absorption of light in water a group of students was blocked by the problem of assigning the correct value to a^0 . According to the students the value was 0, but then reference to the intensity of light at the surface of the water made them change to the value 1, which then was checked on their calculator. And the following from a student's written report shows that the models constructed by the students had both a formal and concrete status:

We saw that cooling of the hot liquid resulted in a decreasing exponential function. Therefore the graph has the form of $y = ba^x + c$. *In the formula the value of c must be the room temperature; else we had to cool the liquid with something colder than room temperature. The graph of heating the ice water was a graph that was opposite to cooling graph. If you place the temperatures from both graphs at the same time you get a horizontal graph.*

In this description the function represented by a formal expression, but it is not considered as a static entity but as a manipulative object that can be transformed. And through the transition to the more formal and generalized model, the extended model does not become detached from the original model (Michelsen, 2001).

5.2 Experiences and perspectives

The experiences from the integrated course are the base for our development of new prototypes of interdisciplinary courses. Recently two master theses resulted in two new courses. Both courses focused on the use of new media to afford students' reasoning and experiencing. One of the courses is a new version of the radioactivity course. The modelling activities are supplemented with a webbased narrative about an accident caused by leakage in a transport of a radioactive substance. The student has to act as a journalist that covers the accident. The other course uses the motion of bicycle wheel to introduce the trigonometric functions. The students videotape the motion of a fixed point on the wheel. The motion of the point is explored and modelled by students' with a software program providing different forms of representations and possibilities for transforming the constructed models.

It's our claim that in learning physics it's not the mathematical formalism that acts as a barrier, too high to overcome for the students – the problem is the missing links between mathematics and physics. Further it is obvious that the isolation problem in mathematics education is transcended when concepts, skills and principles of mathematics and physics – and other subjects – are weaved together into a unified whole. Physics provides mathematics with interesting problems to investigate, and mathematics provides physics with powerful tools in analyzing data and theorizing. The students' interpretation of the mathematical concepts in terms of other disciplines, for instance physics, gives a richer approach and a consistent meaning to the concepts that helps learning. By expanding the domain we are on the track of solving the problem of domain specificity.

6. CONCLUSION

In view of the growth of research in mathematics education over the last decades, it is remarkable that only little attention has been paid to research on the educational relations between mathematics and other subjects. Issues related to this topic are complex, because they comprise two apparently different components, an extra-mathematical and a mathematical context. But if we as mathematics educators take the stance that mathematics has value of solving meaningful problems or even improving society, then we have to design learning environments that are meaningful to and value for the students. In the approach presented in this paper the choice is a model for the curriculum that integrates across the disciplines of mathematics and physics. Integrated means really interdisciplinary and both applied and conceptual mathematics are part of the curriculum.

We do not believe that all mathematics concepts should be developed in an interdisciplinary context. There are topics in mathematic that are the result of structuring within the mathematical system. But we want to emphasize that the more connections that exist among facts, concepts, and procedures, the better the students' understanding, and the more the curriculum is presented as a cohesive program with a range of tentacles, the more likely the students will have a rounded, effective and meaningful education. We do believe that interdisciplinary competences could be the generic methodology that acts as a common denominator for disciplines, such as mathematics and physics. And there is room for considerable improvement. However, an identification and clarification of the interdisciplinary competences is needed as well as further research on the constraints and possibilities for the integration of mathematics and physics.

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