17.8 Nonhomogeneous Linear Equations

We now consider the problem of solving the nonhomogeneous second-order differential equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x). \qquad (*)$$

We assume that two independent solutions, $y_1(x)$ and $y_2(x)$, of the corresponding homogeneous equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

are known. The function $y_h(x) = C_1y_1(x) + C_2y_2(x)$, which is the general solution of the homogeneous equation, is called the **complementary function** for the nonhomogeneous equation. Theorem 2 of Section 17.1 suggests that the general solution of the nonhomogeneous equation is of the form

$$y = y_p(x) + y_h(x) = y_p(x) + C_1y_1(x) + C_2y_2(x),$$

where $y_p(x)$ is any **particular solution** of the nonhomogeneous equation. All we need to do is find *one solution* of the nonhomogeneous equation and we can write the general solution.

There are two common methods for finding a particular solution y_p of the nonhomogeneous equation $(*)$:

1. The method of undetermined coefficients, and
2. The method of variation of parameters.

The first of these hardly warrants being called a *method*; it just involves making an educated guess about the form of the solution as a sum of terms with unknown coefficients and substituting this guess into the equation to determine the coefficients. This method works well for simple DEs, especially ones with constant coefficients. The nature of the *guess* depends on the nonhomogeneous term $f(x)$, but can also be affected by the solution of the corresponding homogeneous equation. A few examples will illustrate the ideas involved.

Example 1 Find the general solution of $y'' + y' - 2y = 4x$.

Solution Because the nonhomogeneous term $f(x) = 4x$ is a first-degree polynomial, we "guess" that a particular solution can be found which is also such a polynomial. Thus we try

$$y = Ax + B, \qquad y' = A, \qquad y'' = 0.$$

Substituting these expressions into the given DE we obtain

$$\begin{aligned} 0 + A - 2(Ax + B) &= 4x \qquad \text{or} \\ -(2A + 4)x + (A - 2B) &= 0. \end{aligned}$$

This latter equation will be satisfied for all x provided $2A+4 = 0$ and $A-2B = 0$. Thus we require $A = -2$ and $B = -1$; a particular solution of the given DE is

$$y_p(x) = -2x - 1.$$

Since the corresponding homogeneous equation $y'' + y' - 2y = 0$ has auxiliary equation $r^2 + r - 2 = 0$ with roots $r = 1$ and $r = -2$, the given DE has the general solution

$$y = y_p(x) + C_1 e^x + C_2 e^{-2x} = -2x - 1 + C_1 e^x + C_2 e^{-2x}.$$

Example 2 Find general solutions of the equations (where $'$ denotes d/dt)

(a) $y'' + 4y = \sin t$,

(b) $y'' + 4y = \sin(2t)$,

(c) $y'' + 4y = \sin t + \sin(2t)$.

Solution

(a) Let us look for a particular solution of the form

$$\begin{aligned} y &= A \sin t + B \cos t \qquad \text{so that} \\ y' &= A \cos t - B \sin t \\ y'' &= -A \sin t - B \cos t. \end{aligned}$$

Substituting these expressions into the DE $y'' + 4y = \sin t$, we get

$$-A \sin t - B \cos t + 4A \sin t + 4B \cos t = \sin t,$$

which is satisfied for all x if $3A = 1$ and $3B = 0$. Thus $A = 1/3$ and $B = 0$. Since the homogeneous equation $y'' + 4y = 0$ has general solution $y = C_1 \cos(2t) + C_2 \sin(2t)$, the given nonhomogeneous equation has the general solution

$$y = \frac{1}{3} \sin t + C_1 \cos(2t) + C_2 \sin(2t).$$

(b) Motivated by our success in part (a), we might be tempted to try for a particular solution of the form $y = A \sin(2t) + B \cos(2t)$, but that won't work, because this function is a solution of the homogeneous equation, so we would get $y'' + 4y = 0$ for any choice of A and B. In this case it is useful to try

$$y = At \sin(2t) + Bt \cos(2t).$$

We have

$$\begin{aligned} y' &= A \sin(2t) + 2At \cos(2t) + B \cos(2t) - 2Bt \sin(2t) \\ &= (A - 2Bt) \sin(2t) + (B + 2At) \cos(2t) \\ y'' &= -2B \sin(2t) + 2(A - 2Bt) \cos(2t) + 2A \cos(2t) \\ &\quad - 2(B + 2At) \sin(2t) \\ &= -4(B + At) \sin(2t) + 4(A - Bt) \cos(2t). \end{aligned}$$

Substituting into $y'' + 4y = \sin(2t)$ leads to

$$\begin{aligned} -4(B + At) \sin(2t) + 4(A - Bt) \cos(2t) + 4At \sin(2t) + 4Bt \cos(2t) \\ = \sin(2t). \end{aligned}$$

Observe that the terms involving $t\sin(2t)$ and $t\cos(2t)$ cancel out and we are left with

$$-4B\sin(2t)+4A\cos(2t)=\sin(2t),$$

which is satisfied for all x if $A=0$ and $B=-1/4$. Hence, the general solution for part (b) is

$$y=-\frac{1}{4}t\cos(2t)+C_1\cos(2t)+C_2\sin(2t).$$

(c) Since the homogeneous equation is the same for (a), (b), and (c), and the nonhomogeneous term in equation (c) is the sum of the nonhomogeneous terms in equations (a) and (b), the sum of particular solutions of (a) and (b) is a particular solution of (c). (This is because the equation is *linear.*) Thus the general solution of equation (c) is

$$y=\frac{1}{3}\sin t-\frac{1}{4}t\cos(2t)+C_1\cos(2t)+C_2\sin(2t).$$

We summarize the appropriate forms to try for particular solutions of constant-coefficient equations as follows:

Trial solutions for constant-coefficient equations

Let $A_n(x)$, $B_n(x)$, and $P_n(x)$ denote the nth-degree polynomials

$$\begin{aligned}A_n(x)&=a_0+a_1x+a_2x^2+\cdots+a_nx^n\\B_n(x)&=b_0+b_1x+b_2x^2+\cdots+b_nx^n\\P_n(x)&=p_0+p_1x+p_2x^2+\cdots+p_nx^n\end{aligned}$$

To find a particular solution $y_p(x)$ of the second-order linear, constant-coefficient, nonhomogeneous DE

$$a_2\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_0y=f(x)$$

use the following forms:

If $f(x)=P_n(x)$	try $y_p=x^mA_n(x)$.
If $f(x)=P_n(x)e^{rx}$	try $y_p=x^mA_n(x)e^{rx}$.
If $f(x)=P_n(x)e^{rx}\cos(kx)$	try $y_p=x^me^{rx}[A_n(x)\cos(kx)+B_n(x)\sin(kx)]$.
If $f(x)=P_n(x)e^{rx}\sin(kx)$	try $y_p=x^me^{rx}[A_n(x)\cos(kx)+B_n(x)\sin(kx)]$.

where m is the smallest of the integers 0, 1, and 2, that ensures that no term of y_p is a solution of the corresponding homogeneous equation

$$a_2\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_0y=0.$$

Resonance

For $\lambda > 0$, $\lambda \neq 1$, the solution $y_\lambda(t)$ of the initial-value problem

$$\begin{cases} y'' + y = \sin(\lambda t) \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

can be determined by first looking for a particular solution of the DE having the form $y = A\sin(\lambda t)$, and then adding the complementary function $y = B\cos t + C\sin t$. The calculations give $A = 1/(1-\lambda^2)$, $B = 0$, $C = (1-\lambda-\lambda^2)/(1-\lambda^2)$, so

$$y_\lambda(t) = \frac{\sin(\lambda t) + (1-\lambda-\lambda^2)\sin t}{1-\lambda^2}.$$

For $\lambda = 1$ the nonhomogeneous term in the DE is a solution of the homogeneous equation $y'' + y = 0$, so one must try for a particular solution of the form $y = At\cos t + Bt\sin t$. In this case, the solution of the initial-value problem is

$$y_1(t) = \frac{3\sin t - t\cos t}{2}.$$

(This solution can also be found by calculating $\lim_{\lambda\to 1} y_\lambda(t)$ using l'Hôpital's rule.) Observe that this solution is unbounded; the amplitude of the oscillations become larger and larger as t increases. In contrast, the solutions $y_\lambda(t)$ for $\lambda \neq 1$ are bounded for all t, though they can become quite large for some values of t if λ is close to 1. The graphs of the solutions $y_{0.9}(t)$, $y_{0.95}(t)$, and $y_1(t)$ on the interval $-10 \leq t \leq 100$ are shown in Figure 17.6.

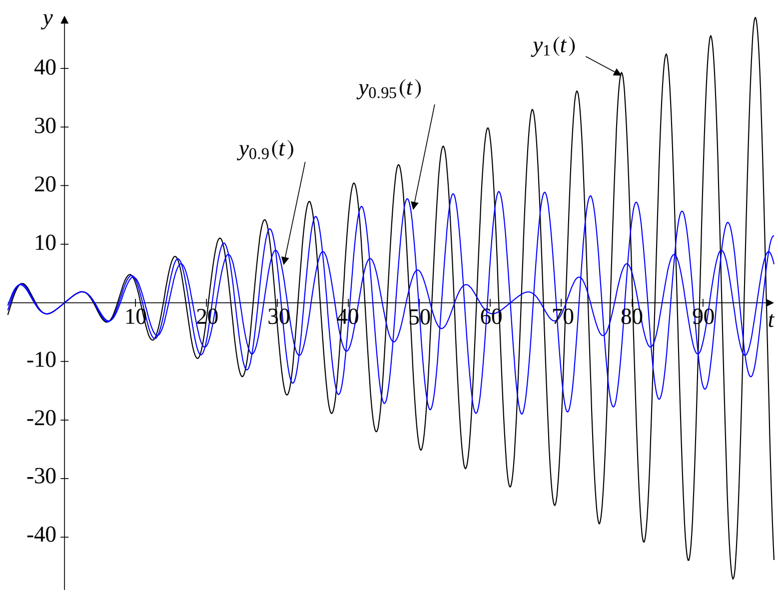

Figure 17.6 Resonance

The phenomenon illustrated here is called **resonance**. Vibrating mechanical systems have natural frequencies at which they will vibrate. If you try to force them to vibrate at a different frequency, the amplitude of the vibrations will themselves vary sinusoidally over time, producing an effect known as **beats**. The amplitudes of the beats can grow quite large, and the period of the beats lengthens as the forcing frequency approaches the natural frequency of the system. If the system has no resistive damping (the one illustrated here has no damping) then forcing vibrations at the natural frequency will cause the system to vibrate at ever increasing amplitudes.

As a concrete example, if you push a child on a swing, the swing will rise highest if your pushes are timed to have the same frequency as the natural frequency of the swing. Resonance is used in the design of tuning circuits of radios; the circuit is tuned (ususally by a variable capacitor) so that its natural frequency of oscillation is the frequency of the station being tuned in. The circuit then responds much more strongly to the signal received from that station than to others on different frequencies.

Variation of Parameters

A more formal method for finding a particular solution $y_p(x)$ of the nonhomogeneous equation when we know two independent solutions, $y_1(x)$ and $y_2(x)$, of the homogeneous equation is to replace the constants in the complementary function by functions, that is, search for y_p in the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Requiring y_p to satisfy the given nonhomogeneous DE provides one equation that must be satisfied by the two unknown functions u_1 and u_2. We are free to require them to satisfy a second equation also. To simplify the calculations below, we choose this second equation to be

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0.$$

Now we have

$$\begin{aligned} y_p' &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' = u_1y_1' + u_2y_2' \\ y_p'' &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''. \end{aligned}$$

Substituting these expressions into the given DE we obtain

$$\begin{aligned} a_2(u_1'y_1' + u_2'y_2') + u_1(a_2y_1'' + a_1y_1' + a_0y_1) + u_2(a_2y_2'' + a_1y_2' + a_0y_2) \\ = a_2(u_1'y_1' + u_2'y_2') = f(x), \end{aligned}$$

because y_1 and y_2 satisfy the homogeneous equation. Therefore u_1' and u_2' satisfy the pair of equations

$$\begin{aligned} u_1'(x)y_1(x) + u_2'(x)y_2(x) &= 0 \\ u_1'(x)y_1'(x) + u_2'(x)y_2'(x) &= \frac{f(x)}{a_2(x)}. \end{aligned}$$

We can solve these two equations for the unknown functions u_1' and u_2' by Cramer's Rule (Theorem 5 of Section 10.6), or otherwise, and obtain

$$u_1' = -\frac{y_2(x)}{W(x)}\frac{f(x)}{a_2(x)}, \qquad u_2' = \frac{y_1(x)}{W(x)}\frac{f(x)}{a_2(x)},$$

where $W(x)$, called the **Wronskian** of y_1 and y_2, is the determinant

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Then u_1 and u_2 can be found by integration.

Example 3 Find the general solution of $y'' - 3y' + 2y = 4x$.

Solution First we solve the homogeneous equation $y'' - 3y' + 2y = 0$, which has auxiliary equation $r^2 - 3r + 2 = 0$ with roots $r = 1$ and $r = 2$. Therefore two independent solutions of the homogeneous equation are $y_1 = e^x$ and $y_2 = e^{2x}$, and the complementary function is

$$y_h = C_1 e^x + C_2 e^{2x}.$$

A particular solution $y_p(x)$ of the nonhomogeneous equation can be found in the form

$$y_p = u_1(x)e^x + u_2(x)e^{2x}$$

where u_1 and u_2 satisfy

$$\begin{aligned} u_1' e^x + 2u_2' e^{2x} &= 4x \\ u_1' e^x + u_2' e^{2x} &= 0. \end{aligned}$$

We solve these linear equations for u_1' and u_2' and then integrate to obtain

$$\begin{aligned} u_1' &= -4xe^{-x} & u_2' &= 4xe^{-2x} \\ u_1 &= 4(x+1)e^{-x} & u_2 &= -(2x+1)e^{-2x}. \end{aligned}$$

Hence $y_p = 4x + 4 - (2x + 1) = 2x + 3$ is a particular solution of the nonhomogeneous equation, and the general solution is

$$y = 2x + 3 + C_1 e^x + C_2 e^{2x}.$$

Remark This method for solving the nonhomogeneous equation is called the **method of variation of parameters.** It is completely general and extends to higher order equations in a reasonable way, but it is computationally somewhat difficult. We could have found y_p more easily had we "guessed" that it would be of the form $y_p = Ax + B$ and substituted this into the differential equation to get

$$\begin{aligned} & -3A + 2(Ax + B) = 4x \\ \text{or} \quad & 2Ax + (2B - 3A) = 4x. \end{aligned}$$

The only way this latter equation can be satisfied for all x is to have $2A = 4$ and $2B - 3A = 0$, that is, $A = 2$ and $B = 3$.

Exercises 17.8

Find general solutions for the nonhomogeneous equations in Exercises 1–12 by the method of undetermined coefficients.

1. $y'' + y' - 2y = 1$

2. $y'' + y' - 2y = x$

3. $y'' + y' - 2y = e^{-x}$

4. $y'' + y' - 2y = e^x$

5. $y'' + 2y' + 5y = x^2$

6. $y'' + 4y = x^2$

7. $y'' - y' - 6y = e^{-2x}$

8. $y'' + 4y' + 4y = e^{-2x}$

9. $y'' + 2y' + 2y = e^x \sin x$

10. $y'' + 2y' + 2y = e^{-x} \sin x$

11. $y'' + y' = 4 + 2x + e^{-x}$ **12.** $y'' + 2y' + y = xe^{-x}$

13. Repeat Exercise 3 using the method of variation of parameters.

14. Repeat Exercise 4 using the method of variation of parameters.

15. Find a particular solution of the form $y = Ax^2$ for the Euler equation $x^2y'' + xy' - y = x^2$, and hence obtain the general solution of this equation on the interval $(0, \infty)$.

16. For what values of r can the Euler equation $x^2y'' + xy' - y = x^r$ be solved by the method of the previous exercise. Find a particular solution for each such r.

17. Try to guess the form of a particular solution for $x^2y'' + xy' - y = x$ and hence obtain the general solution for this equation on the interval $(0, \infty)$.

18. Use variation of parameters to solve $x^2y'' + xy' - y = x$.

19. Consider the nonhomogeneous, linear equation

$$x^2y'' - (2x + x^2)y' + (2 + x)y = x^3.$$

Use the fact that $y_1(x) = x$ and $y_2(x) = xe^x$ are independent solutions of the corresponding homogeneous equation (see Exercise 5 of Section 17.6) to find the general solution of this nonhomogeneous equation.

20. Consider the nonhomogeneous, Bessel equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^{3/2}.$$

Use the fact that $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ are independent solutions of the corresponding homogeneous equation (see Exercise 6 of Section 17.6) to find the general solution of this nonhomogeneous equation.

17.9 Series Solutions

Many of the second-order, linear, differential equations that arise in applications do not have constant coefficients and are not Euler equations. If the coefficient functions of such an equation are sufficiently well behaved we can often find solutions in the form of power series (Taylor series). Such series solutions are frequently used to define new functions, whose properties are deduced partly from the fact that they solve particular differential equations. For example, Bessel functions of order ν are defined to be certain series solutions of Bessel's differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

Series solutions for second-order homogeneous linear differential equations are most easily found near an **ordinary point** of the equation. This is a point $x = a$ such that the equation can be expressed in the form

$$y'' + p(x)y' + q(x)y = 0$$

where the functions $p(x)$ and $q(x)$ are **analytic** at $x = a$. (A function f is analytic at $x = a$ if $f(x)$ can be expressed as the sum of its Taylor series in powers of $x - a$ in an interval of positive radius centred at $x = a$.) Thus we assume

$$p(x) = \sum_{n=0}^{\infty} p_n(x-a)^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n(x-a)^n$$

with both series converging in some interval of the form $a - R < x < a + R$. Frequently $p(x)$ and $q(x)$ are polynomials and so are analytic everywhere. A change of independent variable $\xi = x - a$ will put the point $x = a$ at the origin $\xi = 0$, so we can assume that $a = 0$.

The following example illustrates the technique of series solution around an ordinary point.