Institut for Matematik og Datalogi
Syddansk Universitet

September 5, 2012
JBJ

DM208 – Fall 2012 – Weekly Note 3

Stuff covered in week 36

We gave three different proofs of Menger's theorem (one of them introducing the very important topic of submodularity) and showed a technique based on max-back orderings for computing the edge-connectivity of an undirected graph without using any flow computations. In the exercise part we covered the problems on matroids and greedy algorithms as well as Problem 3 (c)-(e) from the 2011 exam. See notes on matroids at the end of this note.

Lecture September 11, 2012:

The primal-dual algorithm. PS Chapter 5. We will also discuss Sections 11.1 and 11.2 in PS. This is similar to BJG 3.12. See also hand out notes by Bang-Jensen and Toft (handed out at the lecture on August 29).

Lecture September 12, 2012:

Primal-dual algorithms for min-cost flow. PS Chapter 7.

Problems and applications to discuss in the exercise part on September 12,2012:

- Explain how to find (efficiently!) a minimum edge-cut in a graph $G = (V, E)$ using max back orderings (that is, how do find find a cut with $\lambda(G)$ edges?).
- SCH application 1.4.
- Suppose you are given a connected undirected graph $G = (V, E)$ with costs on the edges and your task is to give an algorithm which finds a minimum cost set of $E' \subset E$ edges whose removal disconnects the graph (that is $G - E'$ is not connected). Explain how to do this in polynomial time (hint: use flows).
- SCH exercise 3.2. Hint for (a): consider a maximal matching or apply Hall's theorem.
- SCH Exercise 4.1.
- BJG Exercises 3.33, 3.34 and 3.35.
- SCH application 4.1 be ready to discuss this in the class.
- SCH exercises 3.18, 3.19.

Notes on matroids

Recall that a **base** of a matroid $M = (S, \mathcal{F})$ is a maximal independent set of $\mathcal{F}$.

Theorem 0.1 (Base axioms) *The set $\mathcal{B}$ bases of a matroid $M = (S, \mathcal{F})$ with $\mathcal{F} \neq \emptyset$ satisfy the following axioms:*

(B1) $\mathcal{B} \neq \emptyset$

(B2) $|B_1| = |B_2|$ for all $B_1, B_2 \in \mathcal{B}$.

(B3) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1$ then there exists $y \in B_2$ such that $B_1 - x + y \in \mathcal{B}$.

Proof: It is clear that the bases of M satisfy (B1) and (B2) and (B3) is a special case of the exchange axiom (consider $B_1 - x$ and B_2). $\diamond$

.

The base axioms also define the set of all matroids of a set.

Proposition 0.2 *Let S be a set and let $\mathcal{B} \subseteq 2^S$ be a collection of subsets of S which satisfies (B1)-(B3). Define $\mathcal{F}_\mathcal{B} = \{X \subseteq S | \exists B \in \mathcal{B} : X \subseteq B\}$. Then $M_\mathcal{B} = (S, \mathcal{F}_\mathcal{B})$ is a matorid.*

Proof: Clearly $M_\mathcal{B}$ is s subset system so we just need to show that the exchange axiom holds for $\mathcal{F}_\mathcal{B}$. Let $X, Y \in \mathcal{F}_\mathcal{B}$ with $|Y| = |X| + 1$ and let B_X, B_Y be elements of $\mathcal{B}$ such that $X \subseteq B_X$ and $Y \subseteq B_Y$. Applying (B3) repeatedly we can delete the elements of $B_X - X$ one by one while adding a new element from $B_Y - B_X$ each time. Since $|B_X - X| = |B_Y - Y| + 1$ at some point in this process we have a base B'_X containing X such that the only element of $B_Y - B'_X$ that we can add to $B'_X - w$, $w \notin X$, is an element $y \in Y - X$. Now $B'_X - w + y$ contains $X + y$ so $X + y \in \mathcal{F}_\mathcal{B}$, showing that $Y - X$ contains an element y such that $X + y$ is independent. $\diamond$

Definition 0.3 (dual matroid) *Let $M = (S, \mathcal{F})$ be a matroid with base set $\mathcal{B}$ and rank $r(S) < |S|$. Define $\mathcal{F}^* = \{X | \exists B \in \mathcal{B} : X \cap B = \emptyset\}$. Then $M^* = (S, \mathcal{F}^*)$ is a matroid called the* **dual matroid** *of M.*

Proof: Let $\mathcal{B}^*$ be the set of bases of $\mathcal{F}^*$. We show that $\mathcal{B}^*$ satisfies the base axioms and then it follows from Proposition 0.2 that M^* is a matroid. By definition of $\mathcal{F}^*$, all maximal independent subsets of S have the same size and since $r(S) < |S|$ we have $\mathcal{B}^* \neq \emptyset$ so it only remains to prove that (B3) holds. Let $B_1^*, B_2^* \in \mathcal{B}^*$ and let $x \in B_1^* - B_2^*$ be arbitrary. Note that $(S - B_1^*) \cap (S - B_2^*) + x$ is a subset of $S - B_2^*$ and hence is independent in $\mathcal{F}$. Apply the exchange axiom (in M) to the independent sets $(S - B_1^*) \cap (S - B_2^*) + x$ and $S - B_1^*$ until we have a new base Z of M. This will satisfy $Z = (S - B_1^*) + x - z$ where $z \in (S - B_1^*) \cap B_2^* \subset B_2^*$ so we have shown that we can find $z \in B_2^*$ such that $B_1^* - x + z \in \mathcal{B}^*$. $\diamond$