Institut for Matematik og Datalogi
Syddansk Universitet

September 12, 2012
JBJ

DM208 – Fall 2012– Weekly Note 4

Stuff covered in week 37
The primal-dual algorithm with examples on shortest paths, flows, min-cost flows and the Hitchcock problem (= transportation problem). This was from PS chapters 5 and 7 as well as BJG section 3.4 and 3.10.

Lecture September 18, 2012:
Non-bipartite Matching. SCH 5.1-5.3, see also PS Chapter 10.4-10.5.

Lecture September 19, 2012:

- Tutte's f-factor theorem.
- Arc-disjoint branchings. This is BJG Section 9.5.

Problems and applications to discuss in Week 38:

- SCH application 4.1
- SCH exercise 3.18
- SCH exercise 3.22 (we will probably NOT do this on the blackboard)).
- SCH exercise 3.23.
- Give an example of a graph G with $\nu(G) < \tau(G)$. Argue that for every graph G we have $\tau(G) \leq 2\nu(G)$. Suggest a polynomial algorithm for finding a vertex cover of size at most $2\tau(G)$ in a given graph G.
- Construct an example of a graph G which has two edge-disjoint spanning trees but no matter which max-back ordering you make of G you will not get two edge-disjoint spanning trees by taking the two first max-back forrests F_1, F_2. Hint: there is an example with 4 vertices.
- Show that for every graph $G = (V, E)$ and subsets $X, Y \subseteq V$ we have

$$d(X) + d(Y) \geq d(X \cap Y) + d(X \cup Y) \quad (1)$$
$$d(X) + d(Y) \geq d(X - Y) + d(Y - X) \quad (2)$$

- Show that if G is k-edge-connected and $d(X) = k$, then the subgraph $G[X] = (X, E(X))$ ($E(X)$ is the edges inside X) induced by X has $\lambda(G[X]) \geq \lfloor \frac{k}{2} \rfloor$.

- A graph $G = (V, E)$ is minimally k-edge-connected if $\lambda(G) = k$ but $\lambda(G - e) = k - 1$ for every $e \in E$.
  - Describe an efficient way to find, in a graph G with $\lambda(G) = k$ a spanning subgraph which is minimally k-edge-connected.
  - Show that every minimally k-edge-connected graph G has a vertex of degree k. Hint: consider a set X with $d(X) = k$ and $|X| \geq 2$, observe that it is connected if $k \geq 2$ and then use (1) on the graph obtained by deleting an edge inside X to obtain a smaller set with degree k is G.
- Prove that if a graph is 2-connected (that is, there are at least two internally disjoint (s, t)-paths for every choice of distinct vertices $s, t \in V(G)$), then for every vertex s and edge uv of G there is a cycle C which contains s and the edge uv.
- Show that a graph G has a strongly connected orientation (we replace each edge uv by one of the arcs $u \to v, v \to u$) if and only if G is 2-edge-connected. Also describe an algorithm to find such an orientation or a bad cut.
- Let G be a tree. How many new edges must be added to G to make it 2-edge-connected? Try to construct an algorithm which adds as few edges as possible and try to formulate a min-max result.
- Let $G = (V, E)$ be a k-edge-connected graph and let H be a minimal set of new edges such that $G' = (V, E \cup H)$ is $(k+1)$-edge-connected. Prove that the edges of H form a forest.
- Use max-back orderings to show that every minimally k-edge-connected graph G contains a vertex of degree k (we proved this already but the goal now is to see that this follows from what you have learned about max-back orderings). Extend your argument to also show that in fact there are at least 2 vertices of degree 2. Give an example of a graph (multiple edges are allowed) which is minimaly k-edge-connected and has exactly 2 vertices of degree 2.
- Prove that every minimally k-edge-connected graph has at most $k(n-1)$ edges. Hint: recall what you learned about max-back forrests.