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JBJ

Combinatorial Optimization II (DM209)) — Ugeseddel 2

Handout material in week 5 S. Fortune, J. Hopcroft and J. Wyllie, The directed subgraph homeomorphism problem, Theoretical computer science **10** (1980) 111–121

Stuff covered in week 5:

- Schrijver chapter 8 (see also PS Chapter 13).
- BJG pages 476-482, 482-486 and the hand-out paper by Fortune, Hopcroft and Wyllie on the subgraph homeomorphism problem.
- I also showed how the polynomial algorithm for the k-path problem for acyclic digraphs can be used to solve the directed subgraph homeomorphism problem for acyclic digraphs, that is: Given a fixed acyclic digraph P an input acyclic digraph D and a 1-1 mapping $h : V(P) \rightarrow V(D)$; is it possible to extend h such that arcs of P are mapped to internally vertex disjoint paths in D?
- I gave a proof that the following problem is NP-complete: Given a strongly connected digraph D; does it contain a spanning strongly connected eulerian subdigraph?

Lectures in week 6: Note that there is no lecture monday Feb. 4th so we meet Tuesday and Thursday.

- Schrijver 9.1-9.2.
- Schrijver 9.5
- Possibly also more NP-completeness proofs using the (s,t)-path construction. See the paper "Arc-disjoint spanning sub(di)graphs in Digraphs" on the course page (the proof of Theorem 1.5).
- Exercises:
  - Look at the problems in Proposition 9.2.1. in BJG and show that these problems are equivalent (from a computational point of view, that is, if one if polynomial then so are all the others).
  - Argue that the 2-path problem is NP-complete for digraphs of maximum out-degree (number of arcs out of a vertex) 2. Hint: modify the construction in the NP-completeness proof on pages 480-481 slightly.
  - Give a linear time algorithm for the following problem: Given an acyclic digraph D and distinct vertices $s_1, s_2, \ldots, s_k$ of D, decide whether D contains a path P starting in s_1, ending in s_k and which contains all of the vertices $s_2, s_3, \ldots, s_{k-1}$ in that order. Does the problem become more difficult if we do not require a specific order on $s_2, s_3, \ldots, s_{k-1}$ but just that they are on P?
- 2-Satisfiabilty. Khuller Section 30, BJG Section 1.10.

1 Notes on finding subdivisions for (di)graphs in (acyclic di)graphs

Theorem 1 (Robertson and Seymour, 1995) *For every fixed natural number k there is an algorithm of complexity $O(n^3)$*[1] *for deciding for a given input graph G and distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of G whether G has vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is a (s_i, t_i)-path.*

A **subdivision** of a graph $H = (V_H, E_H)$ in a graph $G = (V, E)$ is a subgraph $G' = (V', E')$ of G and a mapping of H to G' with the property that its is 1-1 on the vertices of H and every edge $e = uv \in E_H$ is mapped to a path P_{uv} from $f(u)$ to $f(v)$ such that every vertex of $P_{uv} - \{f(u), f(v)\}$ has degree 2 in G' (we replace the edge uv by a path in G' and no two paths corresponding to different edges of H intersect except possibly at their ends). This definition also makes sense if H has loops as such a loop at u corresponds to a cycle through $f(u)$ in G'. A subdivision of a digraph is defined analogously.

Corollary 1 *For every graph $H = (V_H, E_H)$ there exists a polynomial algorithm $\mathcal{A}_H$ which for a given input graph $G = (V, E)$ decides whether G contains a subdivision of H.*

[1] The constant here depends heavily on k: the complixity is $O(f(k)n^3)$ where $f(k)$ grows VERY fast in k.

Proof: Let $H = (V_H, E_H)$ be given and assume first that we have fixed a 1-1 mapping $f : V_H \to V$. If there is an edge $uv \in E_H$ such that $f(u)f(v)$ is an edge in G (possibly $u = v$ and then $f(u)f(u)$ is a loop in G), then we can use this edge to realize the path corresponding to the edge uv and consider H minus this edge and G minus the edge $f(u)f(v)$. Hence we can first trim off (select) such pairs and then assume that $f(V_H)$ (the set of images of V_H) is an independent set in G.

Fix an ordering of the edges around each vertex in H: if u has k neighbours then we label these $v_{u,1}, v_{u,2}, \ldots, v_{u,k}$ (notice that the same vertex gets many different labels, one for each of its neighbours in V_H). Clearly for a given edge $e = uv \in E_H$ this gives two labels l_{uv} and l_{vu} (the number it has in u's labelling and in v's labelling). Now consider the graph G_H that we obtain from G by replacing each vertex $f(u)$ by $d_H(u)$ copies, that is, replace u by an independet set $F(u) = \{f(u)^1, f(u)^2, \ldots f(u)^{d_H(u)}\}$ on $d_H(u)$ vertices and join each of these to all neighbours of $f(u)$ in G.

We claim that now G contains a H-subdivision G' where the vertices of H are $\{f(u)|u \in V_H\}$ if and only if G_H contains a collection of disjoint paths $\{P_{uv}|uv \in E_H\}$ where P_{uv} starts in $f(u)^{l_{uv}}$ and ends in $f(v)^{l_{vu}}$. This is easy to see: if the paths exist in G_H then we obtain G' by contracting (identifying) each set $F(u)$ to the single vertex $f(u)$. Conversely, if we are given a subdivision G' of H then we obtain the paths by splitting up each $f(u)$ into $d_H(u)$ distinct vertices. Thus it follows from Theorem 1 that for a fixed 1-1 mapping of $V(H)$ to $V(G)$ we can decide in time $O(n^3)$ whether this mapping extends to a subdivision of H in G. Thus, in polynomial time, we can check for all the $\binom{|V(G)|}{|V(H)|}$ 1-1 mappings of $V(H)$ to $V(G)$ to see whether at least one extends to a homeomorphism of H to G in polynomial time (H is fixed so its size is a constant). $\diamond$

Theorem 2 (Fortune, Hopcroft and Wyllie, 1980) *For any fixed natural number k there exists a polynomial algorithm for deciding whether a given acyclic digraph $D = (V, A)$ with specified vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ has vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is a (s_i, t_i)-path.*

Corollary 2 *For every acyclic digraph $H = (V_H, A_H)$ there exists a polynomial algorithm for deciding whether a given acyclic digraph $D = (V, A)$ contains a subdivision of H.*

Proof: As above it is sufficient to show that we can decide in polynomial time whether a fixed 1-1 mapping of $V(H)$ to $V(D)$ extends to a homoemorphism of H to D so we assume below that a 1-1 mapping of $V(H)$ to $V(G)$ is given.

As above we may assume that the vertices of H are mapped to an independent set in D (if $f(u)f(v)$ is an arc and $uv \in A_H$ then use $f(u)f(v)$ to realize that path and delete the arc uv from A_H. If $f(u)f(v)$ is an arc of D and uv is not and arc of A_H, then we can never use the arc $f(u)f(v)$ in a homeomorphism (because paths must be internally disjoint) and hence we can delete the arc $f(u)f(v)$ from D without changing the problem).

For each vertex $u \in V_H$ fix and ordering of the arcs entering u and an ordering of the arcs leaving u: We label the $d^-_H(u)$ in-neighbours of u $v^-_{u,1}, v^-_{u,2}, \ldots, v^-_{u,d^-_H(u)}$ and we label the $d^+_H(u)$ out-neighbours of u by $v^+_{u,1}, v^+_{u,2}, \ldots, v^+_{u,d^+_H(u)}$. As in the proof above, for a given arc $e = uv \in A_H$ this gives two labels l^+_{uv} and l^-_{uv} (the number it has in u's out-labelling and in v's in-labelling). Given the 1-1 mapping $f : V_H \to V(G)$ we make a new acyclic digraph G_H by replacing each vertex $f(u)$, $u \in V_H$ by two sets $I_{f(u)} = \{v^-_{u,1}, v^-_{u,2}, \ldots, v^-_{u,d^-_H(u)}\}$ and $O_{f(u)} = \{v^+_{u,1}, v^+_{u,2}, \ldots, v^+_{u,d^+_H(u)}\}$ and joing every in-neighbour x of $f(v)$ in G to every vertex y in $I_{f(v)}$ by an arc $x \to y$ and every vertex p of $O_{f(v)}$ to every out-neighbour q of $f(v)$ in G (it is possible that one of the sets $I_{f(v)}, O_{f(v)}$ is empty in which case we add no arcs corresponding to that set).

Now it is easy to show that D contains a subdivision of H if and only if D_H contains vertex disjoint paths $\{P_{uv}|uv \in A_H\}$ where P_{uv} starts in l^+_{uv} and ends in l^-_{uv}. $\diamond$