1 Basic Formulations

Solve the following LP models. Observe for each solution whether it is feasible, infeasible, or unbounded. You can further try to understand the results graphically.

$$
\begin{aligned}
\max\ & 3x_1 + 5x_2 \\
s.t.\ & x_1 \geq 5 \\
& x_2 \leq 10 \\
& x_1 + 2x_2 \geq 10 \\
& x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max\ & x_1 + x_2 \\
s.t.\ & x_1 + 2x_2 \leq 6 \\
& 2x_1 + x_2 \leq 8 \\
& x_1 \geq 7 \\
& x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max\ & x_1 + 2x_2 \\
s.t.\ & x_1 + x_2 \leq 25 \\
& 2x_1 + x_2 \leq 30 \\
& x_2 \leq 35 \\
& x_1\ ,\ x_2 \geq 0
\end{aligned}
$$

2 Maximum Independent Set Problem

Given is a graph $G = (V, E)$ with set of vertices V and set of edges E. The problem is to find a maximum independent set $I \subset V$, i.e., a set of vertices of maximum cardinality such that, for every two vertices in I, there is no edge connecting them. This problem can be formulated as:

$$\max \quad \sum_{i \in V} x_i \tag{1}$$

$$\text{s. t.} \quad x_i + x_j \leq 1 \quad \forall ij \in E \tag{2}$$

$$x_i \in \{0, 1\} \quad \forall i \in V \tag{3}$$

The objective function (1) maximizes the cardinality of the set I, where x_i indicates whether vertice $i \in V$ belongs to I or not. Constraints (2) ensure that adjacent vertices cannot belong to the independent set.

2.1 Problem Instance

The test instance consists of:

- $V = \{1, 2, 3, 4, 5, 6, 7\}$
- $E = \{(1, 2), (1, 3), (2, 4), (2, 5), (3, 4), (3, 6), (4, 5), (4, 6), (5, 7), (6, 7)\}$

3 Assignment Problem

Given are a set of people I and a set of jobs J. The productivity of person $i \in I$ working 1 period at job $j \in J$ is a_{ij}. The problem is to choose an assignment of people to jobs to maximize the total productivity. An assignment is a choice of numbers, x_{ij}, $i \in I$ and $j \in J$, where x_{ij} represents the proportion of time that person i spends on job j. This problem can be formulated as:

$$\max \quad \sum_{i \in I} \sum_{j \in J} a_{ij} x_{ij} \tag{4}$$

$$\text{s. t.} \quad \sum_{j \in J} x_{ij} \leq 1 \qquad \forall i \in I \tag{5}$$

$$\sum_{i \in I} x_{ij} \leq 1 \qquad \forall j \in J \tag{6}$$

$$x_{ij} \geq 0 \qquad \forall i \in I, \forall j \in J \tag{7}$$

The objective function (4) represents the total productivity value that is to maximize. Constraints (5) reflect the fact that a person cannot spend more than 100% of his working time, constraints (6) mean that only one person is allowed on a job at a time, and constraints (7) say that no one can work a negative amount of time on any job.

3.1 Problem Instance

The test instance consists of:

- $I = \{1, 2, 3, 4, 5, 6\}$
- $J = \{1, 2, 3, 4, 5\}$
- Productivity in a $i \times j$ matrix:

$$a = \begin{pmatrix} 10 & 7 & 8 & 9 & 9 \\ 12 & 1 & 2 & 1 & 3 \\ 12 & 7 & 3 & 2 & 3 \\ 4 & 12 & 4 & 4 & 3 \\ 5 & 14 & 1 & 2 & 7 \\ 7 & 13 & 5 & 2 & 7 \end{pmatrix}$$

4 Bin Packing Problem

Given is a set of bins J and a set of items I. Each bin $j \in J$ has a capacity $c_j \geq 0$ attached and each item $i \in I$ has a weight $w_i \geq 0$ attached. The bin packing problem consists of packing items into bins, such that the sum of items in a bin does not exceed the bin's capacity and such that the total number of used bins is minimized. The bin packing problem can be formulated as:

$$\begin{aligned}
\min \quad & \sum_{j \in J} v_j && && (8)\\
\text{s. t.} \quad & \sum_{j \in J} x_{ij} = 1 && \forall i \in I && (9)\\
& c_j v_j - \sum_{i \in I} w_i x_{ij} \geq 0 && \forall j \in J && (10)\\
& x_{ij} \in \{0,1\} && \forall i \in I, \forall j \in J && (11)\\
& v_j \in \{0,1\} && \forall j \in J && (12)
\end{aligned}$$

The objective function (8) minimizes the number of used bins, where $v_j \in \{0,1\}$ indicates if bin j is used. Constraints (9) ensure that each item is packed in exactly one bin. The variable $x_{ij} \in \{0,1\}$ indicates if item $i \in I$ is packed in bin $j \in J$. Constraints (10) make sure that the capacity of a bin is not exceeded by the total weight of items packed into it.

4.1 Problem Instance

The test instance consists of:

- $J = \{1,2,3,4,5\}$
- $I = \{1,2,3,4,5,6,7,8,9,10\}$
- $c = \{25,20,23,27,24\}$
- $w = \{3,6,3,8,2,4,2,3,4,8\}$

5 Facility Location Problem 1

Given is a set of depots N and a set of clients M. There is a fixed cost $f_j \geq 0$ associated with the use of depot j and a transportation cost $c_{ij} \geq 0$ if client i is served by depot j. The problem is to decide which depots to open and which depot serves each client, such that the sum of fixed and transportation costs is minimized. The facility location problem is formulated as:

$$\min \quad \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j \tag{13}$$

$$\text{s. t.} \quad \sum_{j \in N} x_{ij} = 1 \qquad \forall i \in M \tag{14}$$

$$m y_j - \sum_{i \in M} x_{ij} \geq 0 \qquad \forall j \in N \tag{15}$$

$$x_{ij} \in \{0,1\} \qquad \forall i \in M, \forall j \in N \tag{16}$$

$$y_j \in \{0,1\} \qquad \forall j \in N \tag{17}$$

The objective function (13) minimizes the total sum of fixed and transportation costs, where $x_{ij} \in \{0,1\}$ is a variable indicating if client $i \in M$ is served by depot $j \in N$ and where $y_j \in \{0,1\}$ is a variable indicating if depot $j \in N$ is open. Constraints (14) ensure that all clients are served. Constraints (15) make sure that a depot is opened if it serves at least one client. Here m is a constant equal to the number of clients, i.e., $m = |M|$.

5.1 Problem Instance

The test instance consists of:

- $N = \{1,2,3,4\}$
- $M = \{1,2,3,4,5,6\}$
- $f = \{21,16,1,24\}$
- Edge costs in a $m \times n$ matrix:

$$c = \begin{pmatrix} 6 & 2 & 3 & 4 \\ 1 & 9 & 4 & 11 \\ 15 & 2 & 6 & 3 \\ 9 & 11 & 4 & 8 \\ 7 & 23 & 2 & 9 \\ 4 & 3 & 1 & 5 \end{pmatrix}$$

6 Facility Location Problem 2

The facility location problem can also be formulated as:

$$\min \quad \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j \tag{18}$$

$$\text{s. t.} \quad \sum_{j \in N} x_{ij} = 1 \qquad \forall i \in M \tag{19}$$

$$y_j - x_{ij} \geq 0 \qquad \forall j \in N, \forall i \in I \tag{20}$$

$$x_{ij} \in \{0,1\} \qquad \forall i \in M, \forall j \in N \tag{21}$$

$$y_j \in \{0,1\} \qquad \forall j \in N \tag{22}$$

The objective function (18) minimizes the total sum of fixed and transportation costs, where $x_{ij} \in \{0,1\}$ is a variable indicating if client $i \in M$ is served by depot $j \in N$ and where $y_j \in \{0,1\}$ is a variable indicating if depot $j \in N$ is open. Constraints (19) ensure that all clients are served. Constraints (20) make sure that a depot is opened if it serves at least one client.

Note that (15) and (20) differ. The interested reader could think about benefits and drawbacks of the two formulations.

6.1 Problem Instance

The test instance equals that for Facility Location Problem 1.

7 Network Flow Problem

Given is a graph $G = (V, E)$ with a set of vertices V and a set of edges E. Each vertex $i \in V$ has a demand b_i, and the sum of all demands is zero, i.e., $\sum_{i \in V} b_i = 0$. Each edge has attached a cost $c_{ij} \geq 0$ and a capacity $u_{ij} > 0$. The network flow problem consists of finding a feasible flow that satisfies all the demands at minimum cost and can be formulated as:

$$\min \quad \sum_{(ij) \in E} c_{ij} x_{ij} \tag{23}$$

$$\text{s. t.} \quad \sum_{(ij) \in E} x_{ij} - \sum_{(ji) \in E} x_{ji} = b_i \quad \forall i \in V \tag{24}$$

$$0 \leq x_{ij} \leq u_{ij} \quad \forall (ij) \in E \tag{25}$$

The objective function (23) minimizes the total flow cost, where $x_{ij} \geq 0$ is a variable indicating the amount of flow on edge $(ij) \in E$. Constraints (24) ensure that the demand b_i of vertex $i \in V$ is satisfied, and constraints (25) make sure that each variable x_{ij} is non-negative and does not exceed the capacity of edge $(ij) \in E$.

7.1 Problem Instance

The test instance consists of:

- $V = \{1, 2, 3, 4, 5, 6\}$
- $E = \{(1,2), (1,4), (2,3), (3,1), (3,2), (3,5), (3,6), (4,5), (5,1), (5,3), (6,5)\}$
- $b = \{3, 0, 0, -2, 4, -5\}$
- $c = \{8, 5, 4, 3, 8, 6, 2, 5, 6, 4, 3\}$
- $u = \{7, 1, 7, 5, 7, 3, 6, 10, 5, 3, 3\}$

8 Capacity Planning Problem

Given is a digraph $G = (V, E)$ with set of nodes V and set of arcs E. We have W different capacity choices that one can install on the arcs. Each choice $w = 1, \dots, W$ associates a capacity b^w with its cost c^w. The traffic requirements are defined by K oriented pairs of nodes (s^k, t^k), with $s^k, t^k \in V$ and $s^k \neq t^k$, and expected demand d^k of pair $k = 1, \dots, K$. We want to determine the capacity planning and the traffic flows that minimize the total capacity installation cost. Let y_{uv}^w be the binary decision variable indicating the capacity choice $w = 1, \dots, W$ for the arc $uv \in E$. Let x_{uv}^k be the flow variable denoting the amount of traffic routed on the arc $uv \in E$ with respect to the traffic requirement $k = 1, \dots, K$. This problem can be formulated as follows:

$$\min \sum_{uv \in E} \sum_{w=1}^{W} c^w y_{uv}^w \tag{26}$$

$$s.t. \sum_{uv \in E} x_{uv}^k - \sum_{vu \in E} x_{vu}^k = \begin{cases} -d^k, & \text{if } v = s^k, \\ d^k, & \text{if } v = t^k, \\ 0, & \text{otherwise} \end{cases} \qquad \forall v \in V,\ k = 1...K \tag{27}$$

$$\sum_{k=1}^{K} x_{uv}^k \leq \sum_{w=1}^{W} b^w y_{uv}^w \qquad \forall uv \in E \tag{28}$$

$$\sum_{w=1}^{W} y_{uv}^w = 1 \qquad \forall uv \in E \tag{29}$$

$$x_{uv}^k \geq 0 \qquad \forall uv \in E, k = 1...K \tag{30}$$

$$y_{uv}^w \in \{0, 1\} \qquad \forall uv \in E, w = 1...W \tag{31}$$

The objective function (26) represents the total capacity installation cost that is to minimize. The flow conservation property is expressed by (27), guaranteeing that the traffic requirements are entirely fulfilled. Constraints (28) ensure that the available capacity on each link supports all the traffic to be routed through it. Finally, the bandwidth selection is determined by (29).

8.1 Problem Instance

The test instance consists of:

- $V = \{1, 2, 3, 4\}$
- $E = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 1), (4, 3)\}$
- $c = \{5, 8\}$
- $b = \{10, 20\}$
- $(s, t, d) = \{(1, 2, 3), (1, 3, 5), (2, 1, 10), (2, 4, 7), (3, 1, 5), (4, 2, 4), (4, 3, 2)\}$

9 Matching Problem

Given is a graph $G = (V, E)$ with a set of vertices V and a set of edges E. Each edge $e \in E$ has a weight $w_e \geq 0$ attached. The notation $v \in e$ indicates that edge e is incident to vertex v. The problem is to find a matching, i.e., a set of disjoint edges, such that the total sum of edge weights is maximized. The matching problem is formulated as:

$$\max \quad \sum_{e \in E} w_e x_e \tag{32}$$

$$\text{s. t.} \quad \sum_{e \in E: v \in e} x_e \leq 1 \quad \forall v \in V \tag{33}$$

$$x_e \in \{0, 1\} \quad \forall e \in E \tag{34}$$

The objective function (32) maximizes the total sum of edge weights, where $x_e \in \{0, 1\}$ indicates if edge $e \in E$ is part of the solution. Constraints (33) ensure that each vertex has at most one out- and ingoing edge.

9.1 Problem Instance

The test instance consists of:

- $V = \{1, 2, 3, 4, 5\}$
- E consists of edges such that the graph is complete
- Edge weights in a $|V| \times |V|$ matrix:

$$w = \begin{pmatrix} - & 8 & 10 & 2 & 2 \\ 8 & - & 8 & 10 & 6 \\ 10 & 8 & - & 7 & 8 \\ 2 & 10 & 7 & - & 7 \\ 2 & 6 & 8 & 7 & - \end{pmatrix}$$

10 Traveling Salesman Problem

Given is a graph $G = (V, E)$ consisting of a set of vertices V and a set of edges E. Each edge $(ij) \in E$ has an edge cost $c_{ij} \geq 0$ attached. The traveling salesman problem consists of finding a simple path such that each vertex is visited exactly once and such that the traveling salesman returns to his starting point. The goal is to find such a path such that the total sum of edge costs is minimized. The traveling salesman problem is formulated as:

$$
\begin{aligned}
\min \quad & \sum_{(ij)\in E} c_{ij} x_{ij} & & (35)\\
\text{s. t.} \quad & \sum_{j\in V} x_{ij} = 1 & \forall i \in V & \quad (36)\\
& \sum_{j\in V} x_{ji} = 1 & \forall i \in V & \quad (37)\\
& \sum_{(ij)\in E: i,j\in S} x_{ij} \leq |S| - 1 & \forall S \subset V : 2 \leq |S| \leq |V| - 1 & \quad (38)\\
& x_{ij} \in \{0, 1\} & \forall (ij) \in E & \quad (39)
\end{aligned}
$$

The objective function (35) minimizes the total travel cost, where $x_{ij} \in \{0, 1\}$ is a variable indicating if edge $(ij) \in E$ is visited by the traveling salesman. Constraints (36) and (37) ensure that the traveling salesman leaves and enters each vertex. Constraints (38) eliminate sub tours, i.e., that the resulting path consists of a number of small cycles.

10.1 Problem Instance

The test instance consists of:

- $V = \{1, 2, 3, 4, 5, 6\}$
- The graph is complete, hence E consists of edges connecting all pairs of vertices
- Edge costs in a $|V| \times |V|$ matrix:

$$
c = \begin{pmatrix}
- & 9 & 2 & 8 & 12 & 11 \\
9 & - & 7 & 19 & 10 & 32 \\
2 & 7 & - & 29 & 18 & 6 \\
8 & 19 & 29 & - & 24 & 3 \\
12 & 10 & 18 & 24 & - & 19 \\
11 & 32 & 6 & 3 & 19 & -
\end{pmatrix}
$$