First obligatory project in DM85 Spring 2007

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Introduction

The project period formally starts on Friday March 2 and the report must be handed in at the latest Tuesday March 23 after the lecture or to Jørgen Bang-Jensens box in the secretary's room by 14.30. The report must be written in Danish or in English. The project may be solved in groups – permissible group sizes are 1, 2 and 3.

background

A **cut** $(S, V-S)$ in a graph $G=(V,E)$ consist of a partition of V into S and $V-S$ and all edges having one end in S and the other in $V-S$ (such edges are said to be in the cut). The **size** of a cut $(S, V-S)$ is the number of edges in the cut.

A graph $G=(V,E)$ is k-edge-connected if every cut has size at least k.

The problem E2AUG is the following: Given a 2-edge-connected graph $G=(V,E)$, a spanning subgraph $H=(V,F)$ of G and a non-negative cost function c on $E'=E-F$ (corresponding to letting edges of F have cost 0). Find a minimum cost subset $X \subset E'$ (called an **augmentation**) so that the graph $H'=(V,F+X)$ is 2-edge-connected. In the special case when H has no edges we are looking for the cheapest 2-edge-connected spanning subgraph of G.

We shall also consider a special case E1-2AUG of E2AUG in which H is a spanning tree of G. Here a set of edges $X \subset E'$ is called **good** if adding the edges of X to H results in a 2-edge-connected spanning subgraph of G. It is not difficult to see, from the definition of a 2-edge connected graph, that X is good if and only if each edge h of H is **covered** by at least one edge in X, that is, there is some edge $uv \in X$ such that h is an edge of the unique uv path P_{uv} in H.

Question 1.

The following is a natural idea for finding a cheap 2-edge-connected subgraph of a 2-edge-connected graph (that is, we are looking at the case when H has no edges). First find a minimum spanning tree H of G and then solve the E1-2AUG problem for H. Show by a small example that the above heuristic may not always find an optimal solution.

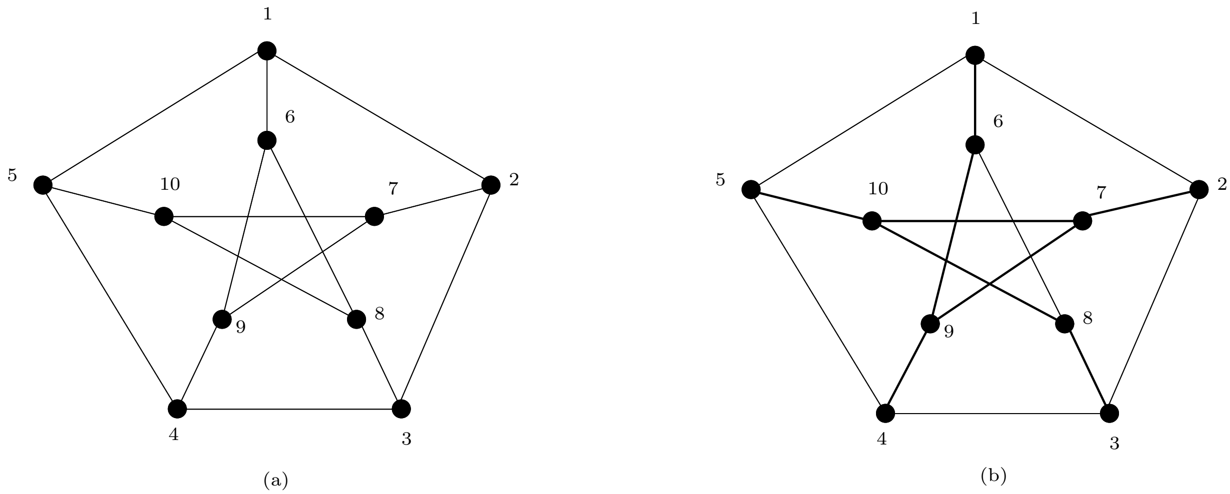

Figure 1: The Petersen graph. In (b) a spanning tree H is shown in bold ($F = \{[1,6],[2,7],[3,8],[4,9],[5,10],[6,9],[7,9],[7,10],[8,10]\}$). All edges not in F have cost 1

Question 2.

(a) Formulate a mathematical model for E2AUG. Use a variable x_e to denote whether the edge $e \in E'$ is to be included in the solution or not.

(b) The Petersen graph is the graph on 10 vertices and 15 edges shown in Figure 1. Prove that the Petersen graph has no Hamiltonian cycle. Hint: you may either use your model and OPL Studio or give a direct mathematical proof. For the latter you may use (without proving it!) that the edges of the Petersen graph cannot be partitioned into 3 perfect matchings (matchings of size 5).

(c) Find an optimum solution for the LP-relaxation of your E2AUG model when G is the Petersen graph and H is the spanning graph with no edges ($H = (V, \emptyset)$). You may do this either by showing a solution and arguing why it is correct and optimal or by using OPL Studio to find it. For the direct proof you may use without proof that the Petersen graph is 3-edge connected.

(d) Discuss the value of the LP-optimum compared to the optimum value of the integer programming formulation of E2AUG with G and H as above. Hint: you can do that without actually solving that problem (but you certainly may do so).

Question 3.

(a) Formulate a mathematical model for E1-2AUG. Hint: in E1-2AUG the only cuts that do not have size at least 2 already via edges in H are the $|V|-1$ cuts that come from deleting an edge of H. That is, S and $V-S$ always denote the two connected components one obtains after deleting an edge from the spanning tree

H. You should try to give an argument for why adding a set of edges that cover all these $|V|-1$ cuts at least once is sufficient to get a good augmentation.

(b) Implement your model in OPL Studio and solve the E1-2AUG problem for the Petersen graph when H is the spanning tree shown in bold in part (b) of Figure 1. Include printouts from OPL studio in your report. You can use the feature "Dump active model and result" from the File menu to do this.

(c) Solve the LP-relaxation, compare with the answer in (b) and discuss your findings.

Question 4.

We now discuss a natural heuristic for solving E1-2AUG. It uses the property that an edge $xy \in E'$ (the edges not in the tree) covers exactly those edges of H which correspond to the path P_{xy} between x and y in H. This suggests the following greedy approach (which we will call **greedy cover**): We start with $X = \emptyset$ and $Z = F$ (all edges of H are uncovered). At any time during the algorithm if $xy \in E'$ covers at least one edge in Z we let $c'(xy)$ be equal to the actual cost $c(xy)$ divided by the number of edges of P_{xy} belonging to Z (the still uncovered edges). If $xy \in E'$ does not cover anything in Z we put $c'(xy) = \infty$. Now the algorithm simply consists of always taking an edge whose c' value is as low as possible, adding this to X, updating Z (by deleting all edges that were covered for the first time by xy) and then updating c'. This step is repeated until $Z = \emptyset$.

(a) Illustrate the heuristic greedy cover on the Petersen graph with H as the spanning tree in Figure 1 (b) and all edge costs equal to one.

(b) Show by an example based on the Petersen graph, the spanning tree in part (b) of Figure 1 and a proper choice of costs for the edges in E' (the non-bold ones) that the heuristic does not always find the optimum solution. Can you say anything about how bad it could perform?

Question 5.

Your boss informs you that you will be put in charge of the rather large project of implementing the intra-net and the corresponding organizational changes in the company. You immediately remember something about project management from your university studies. To get into the tools and the way of thinking, you decide to solve the following test case using PERT/CPM:

Find the duration of the project if all activities are completed according to their normal-times.

In Hillier and Lieberman Section 10.5, a general LP-model is formulated allowing one to find the cheapest way to shorten a project. Formulate the model for the test project and state the dual model.

In the LP-model, there is a variable x_i for each activity i in the project. There are also two constrains, "$x_i \geq 0$" and "$x_i \leq D_i - d_i$". Denote the dual variables of the latter g_i. Argue that if $d_i < D_i$, then either g_i is equal to 0 or $x_i = D_i - d_i$ in any optimal solution.

Activity	Imm. pred.	Normal-time (D_i)	Crash-time (d_j)	Unit cost when shortening
e_1		3	1	3
e_2		4	2	4
e_3	e_1	4	2	1
e_4	e_1, e_2	6	1	3
e_5	e_1	5	5	0
e_6	e_3, e_4	4	4	0

Table 1: Data for the activities of the test case.

What is the additional cost of shortening the project time for the test case to 12?