Obligatory project in DM515 Spring 2011

Jørgen Bang-Jensen

Startdate: April 26, 2011

Introduction

The project period formally starts on Tuesday April 26 and the report must be handed in to the instructor at the latest Wednesday May 11 after the exercise classes or to Alessandro's box in the secretary's room by 10.30 May 11. **The report must be written in English**. The project may be solved in groups – permissible group sizes are 1, 2 and 3. I strongly encourage group sizes of 3. Reports in LateX are prefered, but you do not have to make the drawings in XFIG or similar. I suffices to make a nice hand drawing and include it in the report.

Background

A **cut** $(S, V-S)$ in a graph $G=(V,E)$ consist of a partition of V into S and $V-S$ and all edges having one end in S and the other in $V-S$ (such edges are said to be in the cut). The **size** of a cut $(S, V-S)$ is the number of edges in the cut.

A graph $G=(V,E)$ is k-edge-connected if every cut has size at least k.

The problem E2AUG is the following: Given a 2-edge-connected graph $G=(V,E)$, a spanning subgraph[1] $H=(V,F)$ of G and a non-negative cost function c on $E' = E \backslash F$ (corresponding to letting edges of F have cost 0). Find a minimum cost subset $X \subset E'$ (called an **augmentation**) so that the graph $H' = (V, F+X)$ is 2-edge-connected. In the special case when H has no edges we are looking for the cheapest 2-edge-connected spanning subgraph of G (we call this problem MS2EC, se the model for this below).

We shall also consider a special case of E2AUG in which H is a spanning tree of G. We give this special case the name E1-2AUG. When we consider the problem E1-2AUG, a set of edges $X \subset E'$ is called **good** if adding the edges of X to H results in a 2-edge-connected spanning subgraph of G. It is not difficult to see, from the definition of a 2-edge connected graph, that X is good if and only if each edge h of H (which is a spanning tree when we consider E1-2AUG) is **covered** by at least one edge in X, that is, there is some edge $uv \in X$ such that h is an edge of the unique uv path P_{uv} in H.

All the problems MS2EC, E1-2AUG and E2AUG are NP-hard as optimization problems. In particular MS2EC generalizes the hamiltonian cycle problem: A graph $G=(V,E)$ with n vertices has a hamiltonian cycle if and only if the optimum solution to the instance G, $c \equiv 1$ of MS2EC has value n. This already implies that E2AUG is NP-hard because MS2EC is a special case of E2AUG as explained above. Intuitively finding good solutions (close to optimal) for E1-2AUG should be easier than the full E2AUG problem since in E1-2AUG we just have to cover

[1] This means that H contains all the vertices of G and that F is a subset of E.

all edges of the spanning tree H (we have a structure to guide us). However, it can be shown that E1-2AUG is also NP-hard.

A mathematical model for MS2EC

Let $G = (V, E)$ be a 2-edge-connected graph with a cost function $c : E \to \mathbf{R}$ We use a 0-1 variable x_e to denote whether or not an edge $e \in E$ should be included in the solution. Now the formulation is as follows

$$\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e && (1)\\
\text{s. t.} \quad & \sum_{\{e \in E : |e \cap S| = 1\}} x_e \geq 2 \quad \forall S \subset V : 1 \leq |S| \leq |V| - 1 && (2)\\
& x_e \in \{0, 1\} \qquad \forall (ij) \in E && (3)
\end{aligned}$$

The condition (2) says that whenever we consider a non-empty proper subset S of V there must be at least two edges having precisely one end vertex in S and the other in $V \setminus S$ (these are edges with $|e \cap S| = 1$). This is equivalent to saying that every cut has size at least 2.

Question 1.

The following is a natural idea for finding a heuristic solution to MS2EC. First find a minimum spanning tree H of G and then solve the E1-2AUG problem for H. Show by a small example that the above heuristic for MS2EC may not always find an optimal solution. That is, construct an example of a 2-edge-connected graph G and a weight function c on its edges so that no optimal solution for that instance of MS2EC contains all the edges of a minimum spanning tree in G.

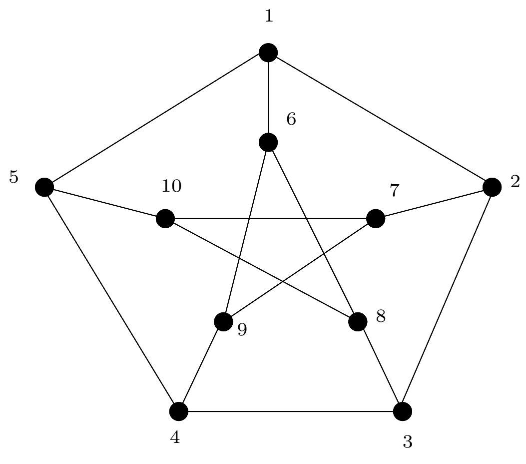
Question 2.

(a) Formulate a mathematical model for E2AUG. Use a variable x_e to denote whether the edge $e \in E'$ is to be included in the solution or not. Hint: look at the model for MS2EC and modify it so that you get a model for E2AUG. Recall that MS2EC corresponds to the case when $F = \emptyset$, so you should just find out how to make the edges from F contribute in (2)

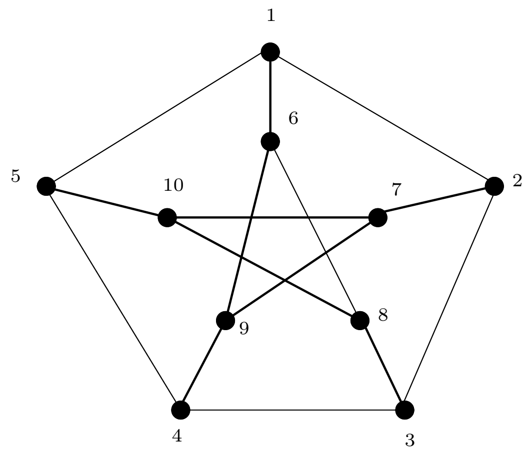
(b) The Petersen graph is the graph on 10 vertices and 15 edges shown in Figure 1. Prove that the Petersen graph has no Hamiltonian cycle. Hint: you may either use your model (or the one above for MS2EC) and Zimpl/cplex or give a direct mathematical proof. For the latter you may use (without proving it!) that the 15 edges of the Petersen graph cannot be partitioned into 3 perfect matchings (matchings of size 5).

(c) Find an optimum solution for the LP-relaxation of your E2AUG model when G is the Petersen graph and H is the spanning graph with no edges ($H = (V, \emptyset)$). You may do this either by showing a solution and arguing why it is correct and optimal or by using Zimpl/cplex to find it. For the direct proof you may use without proof that the Petersen graph is 3-edge connected.

(d) Discuss the value of the LP-optimum compared to the optimum value of the integer programming formulation of E2AUG with G and H as above. Hint: you can do that without actually solving that problem (but you certainly may do so).



(a)



(b)

Figure 1: The Petersen graph. In (b) a spanning tree H is shown in bold ($F = \{[1,6],[2,7],[3,8],[4,9],[5,10],[6,9],[7,9],[7,10],[8,10]\}$). All edges not in F have cost 1

Question 3.

(a) Formulate a mathematical model for E1-2AUG. Hint: in E1-2AUG the only cuts that do not have size at least 2 already via edges in H are the $|V|-1$ cuts that come from deleting an edge of H. That is, S and $V-S$ always denote the two connected components one obtains after deleting an edge from the spanning tree H. You should try to give an argument for why adding a set of edges that cover all these $|V|-1$ cuts at least once is sufficient to get a good augmentation.

(b) Implement your model in Zimpl and solve the E1-2AUG problem for the Petersen graph when H is the spanning tree shown in bold in part (b) of Figure 1. Include printouts from Zimpl and Cplex in your report.

(c) Solve the LP-relaxation, compare with the answer in (b) and discuss your findings.

Question 4.

We now discuss a natural heuristic for solving E1-2AUG. It uses the property that an edge $xy \in E'$ (the edges not in the tree) covers exactly those edges of H which correspond to the path P_{xy} between x and y in H. This suggests the following greedy approach (which we will call **greedy cover**): We start with $X = \emptyset$ and $Z = F$ (all edges of H are uncovered). At any time during the algorithm if $xy \in E'$ covers at least one edge in Z we let $c'(xy)$ be equal to the actual cost $c(xy)$ divided by the number of edges of P_{xy} belonging to Z (the still uncovered edges). If $xy \in E'$ does not cover anything in Z we put $c'(xy) = \infty$. Now the algorithm simply consists of always taking an edge whose c' value is as low as possible, adding this to X, updating Z (by deleting all edges that were covered for the first time by xy) and then updating c'. This step is repeated until $Z = \emptyset$.

(a) Illustrate the heuristic greedy cover on the Petersen graph with H as the spanning tree in Figure 1 (b) and all edge costs equal to one. You should show what happens in each step.

(b) Show by an example based on the Petersen graph, the spanning tree in part (b) of Figure 1 and a proper choice of costs for the edges in E' (the non-bold ones) that the heuristic does not always find the optimum solution.

Question 5.

Activity	Imm. pred.	Normal-time (D_i)	Crash-time (d_j)	Unit cost when shortening
e_1		3	1	3
e_2		4	2	4
e_3	e_1	4	2	1
e_4	e_1, e_2	6	1	3
e_5	e_1	5	5	0
e_6	e_3, e_4	4	4	0

Table 1: Data for the activities of the test case.

Your boss informs you that you will be put in charge of the rather large project of implementing the intra-net and the corresponding organizational changes in the company. You immediately remember something about project management from your university studies. To get into the tools and the way of thinking, you decide to solve the following test case using PERT/CPM:

(a) Find the duration of the project if all activities are completed according to their normal-times.

In Clausen and Larsen (notes on homepage of course) Section 5.5 a general LP-model is formulated allowing one to find the cheapest way to shorten a project.

(b) Formulate the LP-model for the test project and state the dual model. Explain in a few words on how you derive the dual model.

(c) What is the additional cost of shortening the project time for the test case to 12?