Institut for Matematik og Datalogi
Syddansk Universitet

April 12, 2011
JBJ

DM515 – Spring 2011 – Weekly Note 3

NB: Switch of classes:

- There will be exercise classes on Monday April 18, 12-14 in U20
- There will be Lecture on Wednesday April 27 8-10 in U140
- There will be exercises on Thursday April 28 12-14 in U20

Obligatory Assignment :

This is available by the end of week 15 on the homepage of the course. As the name indicates you must hand in a satisfactory solution to this assignment in order to be allowed to attend the exam in June.

You may (and should!) work in groups of up to 3 on these problems and you must hand in your answer (one per group) to Alessandro on May 11 at the exercises or by putting your solutions in his mailbox at the secretarys office. Note that there will be no second round of handing in solutions so you have to get it right the first time!

We do not insist on 100% correct solutions in order for you to have the assignment approved but it must be clear from your report that you have worked carefully on the problems.

Stuff covered in Week 15:

- MG Sections 4.1-4.2.
- MG Sections 5.1-5.3.

Exercises April 18 in U20:

- Questions 3.4.1-3.4.7 in Gutin's notes (see homepage of course). Not all of these will be covered at the exercises as at least 1 hour will be spent on the exercises below.
- **Valid in-equalities for integer programming problems**. Suppose we are given an integer program (IP): $\max\{cx : x \in X\}$, where $X = \{x : Ax \leq b, x \in \mathcal{Z}^n_+\}$ is the set of integer points in $\mathcal{R}^n$ which satisfy $Ax \leq b$ (where A is an $m \times n$ matrix and

$b \in \mathcal{R}^m$. We say that the inequality $rx \leq r_0$ (where $r \in \mathcal{R}^n$ and $r_0 \in \mathcal{R}$) is **valid** for X if $rx \leq r_0$ holds for every $x \in X$. That is, by adding this inequality to the problem (IP) we do not change the set of feasible solutions (note that we may exclude several non-integer solutions, which is exactly what we want to do!!). The definition extends to mixed integer programming problems with the obvious extension of the definition.

1. Consider the maximum weight matching problem from MG page 33 and recall that the formulation (3.1) also works for the general case when $G = (V, E)$ is not necessarily bipartite. Here $X = \{x \in \{0,1\}^m : \sum_{\{v:uv \in E\}} x_{uv} = 1 \text{ for all } u \in V\}$, where $m = |E|$ and the objective is to maximize the objective $\sum_{uv \in E} w_{uv} x_{uv}$. Argue that for every subset $T \subseteq V$ with $|T| \equiv 1$ modulo 2 and $|T| \geq 3$, the following is a valid inequality with respect to the set X:

$$\sum_{uv \in E(T)} x_{uv} \leq \frac{|T|-1}{2},$$

where $E(T) = \{uv \in E : u, v \in T\}$.

2. Suppose below that $X = \{x \in \mathcal{Z}_+^n : Ax \leq b\}$, A is an $m \times n$ matrix with columns $a_1, a_2, \ldots, a_m$ and $u \in \mathcal{R}_+^m$ is a vector of coefficients.

    (i) Argue that the following inequality is valid for X:

$$\sum_{j=1}^{n} \sum_{i=1}^{m} u_i a_{ij} x_j \leq \sum_{j=1}^{m} u_j b_j \tag{1}$$

    This says that taking any non-negative combination of the original inequalities gives a valid inequality.

    (ii) Argue that the following inequality is valid for X:

$$\sum_{j=1}^{n} \lfloor \sum_{i=1}^{m} u_i a_{ij} \rfloor x_j \leq \sum_{j=1}^{m} u_j b_j \tag{2}$$

    Argue that the following inequality is valid for X:

$$\sum_{j=1}^{n} \lfloor \sum_{i=1}^{m} u_i a_{ij} \rfloor x_j \leq \lfloor \sum_{j=1}^{m} u_j b_j \rfloor \tag{3}$$

    The 3 step procedure above is called the **Chvàtal-Gomory procedure** and it can be shown that every valid inequality of an integer program can be constructed using these simple steps.

- **Gomory cuts**. Consider the integer program (IP) $z_{IP} = \max\{cx : Ax = b, x \geq 0 \text{ and integer}\}$ and its LP-relaxation (LP) $z_{LP} = \max\{cx : Ax = b, x \geq 0\}$. Suppose that we have solved (LP) using the Simplex method. If all basic variables are integers then we have solved (IP) so suppose this is not the case. Choose a basic variable x_{B_i} so that x_{B_i} is not an integer in the optimal solution to (LP). Since x_{B_i} can be read off the last Simplex tableau, in that tableau we have expressed x_{B_i} as follows (recall MG page 65):

$$x_{B_i} = p_i + \sum_{j=1}^{n-m} Q_{ij} x_{N_j}, \tag{4}$$

where $x_{N_1}, \ldots, x_{N_{n-m}}$ are the non-basic variables and $p_i > 0$ is not an integer.

(I) Show that the following inequality is valid for (IP):

$$x_{B_i} + \sum_{j=1}^{n-m} \lfloor -Q_{ij} \rfloor x_{N_j} \leq \lfloor p_i \rfloor. \tag{5}$$

Hint: isolate p_i and use that in any solution to (IP) all variables are non-negative integers.

(II) Show that the following is a valid inequality for (IP):

$$\sum_{j=1}^{n-m} (-Q_{ij} - \lfloor -Q_{ij} \rfloor) x_{N_j} \geq (p_i - \lfloor p_i \rfloor). \tag{6}$$

(III) Argue that the optimal LP solution given by $x_{B_1}, \ldots, x_{B_m}$ does not satisfy the inequality (6). Hint: the right hand side is larger than zero.

Thus adding this to the formulation of (IP), giving a new equivalent formulation (IP'), eliminates the current optimal LP solution and hopefully tightens the upper bound provided by the LP-relaxation of the extended problem (IP'). This procedure of adding valid inequalities to an integer programming formulation and thereby improving the bound provided by the LP relaxation is known as the **cutting plane method**. Eventually, when we have added enough of these cuts we reach the so-called **ideal formulation** where all extreme points are integer points and now solving the corresponding LP gives the optimum integer solution.

- Consider the integer program (IP)

$$\begin{aligned} z = \max 4x_1 - x_2 & \\ 7x_1 - 2x_2 &\leq 14 \\ x_2 &\leq 3 \\ 2x_1 - 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \text{ and integer} \end{aligned}$$

1. Convert the integer program to one in equality form by adding slack variables and observe that these variables must also be integers since all data is integer.
2. Solve the LP-relaxation of this new IP in equality form by the Simplex method.
3. Identify a basic variable which is not an integer and use the procedure above to construct a Gomory cut which when added to the IP formulation will eliminate the current optimal LP solution.
4. If you have the energy, then try to make a few more rounds of solving the LP, finding a Gomory cut, adding it and resolving.

Lecture April 27 in U140:

- Some remarks on the rest of Chapter 5 on the Simplex method.
- Gutin Chapter 4 and Section 6.1-6.3 in MG. See also Chapter 2 in Clausen and Larsen.

Exercises April 28 in U20:

1. Summer exam 2008 Problem 1 (a)-(c)
2. Summer exam 2008 Problem 3
3. The **uncapacitated facility location problem** (UFL) is as follows:

   Given a set $N = \{1, 2, \dots, n\}$ of potential depots and $M = \{1, 2, \dots, m\}$ of clients, suppose there is a fixed cost f_j associated with the use of depot j (opening it etc.) and a transportation cost c_{ij} if all of client i's order is delivered from depot j. The problem now is to determine which depots to open and which depot serves each client so as to minimize the sum of the fixed costs and the transportation costs.

   (a) Argue that the following models the problem. Here $x_{ij} \in [0, 1]$ is the fraction of the demand of client i which is satisfied by depot j and $y_j = 1$ if depot j is opened and $y_j = 0$ otherwise.

$$z_{UFL} = \min \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} f_j y_j$$

such that

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i \in M \tag{7}$$

$$\sum_{i \in M} x_{ij} \leq m y_j \text{ for all } j \in N \tag{8}$$

$$y_j \in \{0, 1\} \text{ for all } j \in N \tag{9}$$

$$0 \leq x_{ij} \leq 1 \text{ for all } i \in M, j \in N \tag{10}$$

(b) Explain why we can replace the condition in (8) by the following and still have the same set of feasible solutions.,

$$x_{ij} \leq y_j \text{ for all } i \in N, j \in M \tag{11}$$

(c) Now consider the LP-relaxation of these two versions of UFL (using (8) respectively, (11)), where we replace the condition $y_j \in \{0, 1\}$ by the condition $0 \leq y_j \leq 1$ for each $j \in N$.

i. Argue that for both versions the optimum solution to the LP-relaxation is a lower bound for Z_{UFL}.

ii. Show that every feasible solution to the LP-relaxation of the second formulation (using (11) is also a feasible solution to the LP-relaxation of the first formulation (using (8)).

iii. Suppose that n divides m, so $m = kn$ for some integer $k \geq 2$. Construct a solution to the LP-relaxation of the first formulation which is not a feasible solution to the LP-relaxation of the second formulation. Hint: let each depot serve k clients.

iv. Which is the two models for UFL would you consider the better in terms of using the LP-relaxation to get information about integer (in y variables) solutions to UFL?