Institut for Matematik og Datalogi
Syddansk Universitet

May 18, 2011
JBJ

DM515 – Spring 2011 – Weekly Note 7

Stuff covered in Week 20:

- MG sections 8.2-8,3
- Overview of the course
- Hints for the exam

Note that there are no lectures in week 21, but 3 exercise classes!
These are

- Monday, May 23 12-14 in U20
- Wednesday 8-10 in U140
- Thursday 12-14 in U20

Problems for the exercises in week 21:

1. Read Section 2.3.4 in BG on the Bellman-Ford algorithm and be prepared to discuss the correctness of the algorithm. Just as in Dijkstra's algorithm we can maintain, for every vertex $v \neq s$ a predecessor for v on the current shortest path from s to v. These start out being "nil" and when $d(v)$ is changed by relaxing the arc uv the predecessor will become u. Show how to use the predecessor arcs to find a negative cycle in the case the step 3 (bottom of page 56) returns the message that D has such a cycle. Hint: consider what happens when the predecessor graph contains a cycle for the first time (its starts having no arcs).

2. Questions 6.5.2 and 6.5.9 in Gutin's notes.

3. Another formulation of TSP. Let x_{ij} be a 0-1 variable indicating whether or not vertex j comes immediately after vertex i in the tour (that is, we fix an orientation of the tour so if i is just after j, then j is not just after i, i.e. at most one of x_{ij}, x_{ji} can be 1) and let c_{ij} be the distance. The length of the tour given by x is then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{1}$$

which we wish to minimize over all x which correspond to a tour (a hamiltonian cycle). Since each vertex is preceeded and followed by exactly one vertex in a tour x must satisfy (2) and (3):

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n \tag{2}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n \tag{3}$$

The optimal solution to (1)-(3) may still not be a hamiltonian cycle, but it is always a collection of cycles. In fact (1)-(3) describe exactly the assignment problem which you have seen in BG 3.12. In order to force the solution to be just one cycle we add the following sets of conditions:

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1, \text{ for every proper subset } S \subset \{1, 2, \ldots, n\} \tag{4}$$

The problem about the formulation above is the there are exponentially many constraints. Now we will look at another formulation (due to Miller, Tucker and Zemlin 1960) which also eliminates subtours and has only a polynomial number of constraints.

(a) Fix vertex 1 to be the home base and for each other vertex i let u_i be an arbitrary real number. Show that if x is a feasible solution to (1)-(4), then we can choose values for $u_2, u_3, \ldots, u_n$ so that the following holds:

$$u_i - u_j + n x_{ij} \leq n - 1, \quad i, j = 2, 3, \ldots, n. \tag{5}$$

Hint: consider the number of edges from vertex 1 to vertex i along the tour corresponding to x and choose u_i based on this.

(b) Show that if x is a 0-1 solution satisfying (2), (3) but violating (4), then (5) cannot hold for all $i, j = 2, 3, \ldots, n$. Hint: consider the sum of these equations along a subtour which does not contain vertex 1.

(c) The observation above shows that (1),(2), (3), (5) and x 0-1 valued is a valid formulation of TSP. Discuss the quality of the LP-relaxation of this formulation compared to the classical one using the subtour constraints.

4. Consider the fractional LP solution to a TSP problem in Figure 1. Find a valid in-equality (one which holds for all 0-1 solutions) which cuts off the LP solution x^* of Figure 1. Hint: Does x^* satisfy the constraints (4)?

5. Consider the fractional LP solution y^* to a TSP problem given in Figure 2. Identify a violated Comb inequality, which when added to the formulation will cut away y^*.

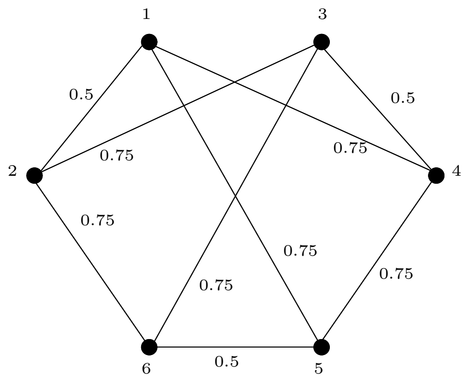

Figure 1: Fractional solution to a TSP problem.

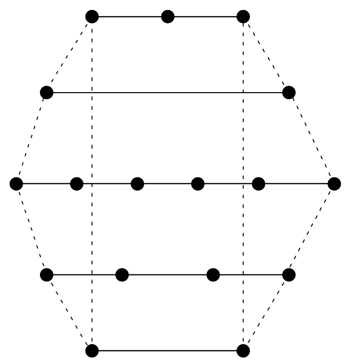

Figure 2: A fractional LP solution y^* to a TSP instance. Dotted lines mean $y* = \frac{1}{2}$ and full lines mean $y^* = 1$.

6. Summer 2008 Problem 4.

7. Consider the following instance of the scheduling problem from Section 8.3 in MG: There are 2 machines $M = \{1, 2\}$ and 4 jobs $J = \{3, 4, 5, 6\}$. The processing times of the jobs are given by $d_{13} = 5, d_{23} = 3, d_{14} = 5, d_{24} = 2, d_{15} = 10, d_{25} = 12, d_{16} = 6, d_{26} = 8$.

   (a) Formulate the problem of minimizing the makespan as an integer programming problem in the variables x_{ij},$i \in \{1, 2\}, j \in \{3, 4, 5, 6\}$ and t. That is, write out the full integer programming problem.

   (b) Solve the problem by inspection .

   (c) Consider the LP-relaxation and show that the following solution is feasible (only non-zero values are shown: $x_{23} = x_{24} = x_{16} = 1$ and $x_{15} = x_{25} = 1/2$. It can be shown that this is an optimal solution (you may check this using the computer software if you wish). Use the method of Section 8.3 to obtain an integer solution from this LP solution. Discuss the value of this solution compared to the optimal IP solution.

   (d) Find the dual of the LP-relaxation above (convert primal to a maximization problem first).

   (e) Use this Dual to find a good bound on the optimal value of the LP problem (guess a solution).

8. Consider again the scheduling problem from Section 8.3 in MG. Suppose that a given subset $J' \subseteq J$ of jobs must be scheduled on the same machine (it can be any of the machines and on this machine they may be scheduled in any of the possible orders). Show how to change the model to handle this case.

9. Clausen and Larsen Section 9.5 Exercises 5, 8

10. DM85 spring 2007 project 2: Problems 1, 2, 3. This is available on the bottom of the home page of the course.

Final pensum

- Matousek and Gaertner: Understanding and using linear programming, Springer Verlag, Berlin, 2006. Pages 1-48, 53-89 (there will be no exam questions in Sections 5.8, 5.9) and 142-156, 204 (complementary slackness).

- J. Bang-Jensen and G. Gutin, Digraphs: Theory algorithms and applications, Springer Verlag, London 2001. Pages 55-58, 95-116, 128-134, 137-140.

- G. Gutin, Computational Optimisation, Notes from Department of Computer Science, Royal Holloway, University of London. pages 1-15, 22-24 og 32-60.

- J. Clausen og J. Larsen, Supplementary notes to networks and integer programming, DTU 2009. Pages 5-18, 69-92 og 143-162.
- W.J. Cook, W. H. Cunningham, W.R. Pulleyblank og A. Schrijver, Combinatorial optimization, Wiley 1998. Siderne 261-265. This is the section about cutting planes for TSP og Comb-inequalities. These pages (pages 252-271) were handed out at the lectures (several time). In case you still (!) don't have a copy get one from the library or ask Mette for a copy.
- All weekly notes and material on these. In particular you should know how to use all methods discussed on the weekly notes.

The exam June 6 9-13:

You are not allowed to use a computer but books notes, pocket calculator is allowed. **It is very important that you bring all material in the pensum list to the exam including weekly notes!!**

At the exam you should be able to formulate simple IP and LP problems from descriptions in words or argue that a proposed model is correct for the problem described. You should be able to apply the methods covered in the course (such as Gomory cuts, Branch and Bound, adding valid inequalities, comb inequalities, reducing to and solving a flow problem, etc). Most of the exam will check that you understand the methods and models and that you can use these.

You should be able to use things such as the max flow min cut theorem, (LP) relaxation (such as IP to LP or TSP to the assigment problem). you must be able to derive the dual problem of an LP problem and apply the duality theorem. The above is by no means an exhaustive list (see the competency list for the course) but it should give you a feeling what is expected from you. **Everyone who worked well during the course and who can apply the methods will be able to do well at the exam.**