Institut for Matematik & Datalogi
Syddansk Universitet

December 3, 2010
JBJ

DM528: Combinatorics, Probability and Randomized Algorithms — Ugeseddel 5

Lecture Monday December 6, 2010:

- Kleinberg and Tardós: Sections 13.1-13.3. This is in the red notes

Lecture Friday December 10, 2010:

- Kleinberg and Tardós: Sections 13.4-13.5, 13.9.

Exercises Wednesday December 8, 2010:

- Left over exercises from previous weekly notes.
- Section 7.5: 6, 8, 14, 24, 26.
- Section 7.6: 2, 4, 6, 22
- 2003.13.5 page 2
- 2005.06.6 page 12. Hint: You can use indicator random variables to estimate the number of times a bit shifts.

Second obligatory assignment Has been available on the course page since November 29.

Notes on inclusion-exclusion.

- A probability version of inclusion-exclusion:

  Theorem 0.1 *Let $A_1, A_2, \ldots, A_n$ be events in a probability space and denote by S_k the following sum*

  $$S_k = \sum_{1 \le i_1 < i_2 \ldots < i_k \le n} p(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}).$$

  Then we have

  $$p(A_1 \cup A_2 \cup \ldots \cup A_n) = S_1 - S_2 + \ldots + (-1)^{n-1} S_n$$

  Proof: We give a proof using indicator random variables. For a given event A and element s in the sample space, the random variable $I_A(s)$ equals 1 if $s \in A$ (e.g. If S is the set of outcomes of rolling a pair of dice, then s is a possible outcome, say $s = (3,5)$, so if A_1 is the event that one of the die show an even number and A_2 the event that the sum of the dices is even, then $I_{A_1}(s) = 0$ and $I_{A_2}(s) = 1$). The usefulness of indicator random variables comes from the fact that $E(I_A) = p(A)$. This is seen as follows:

  $$E(I_A) = \sum_{s \in S} I_A(s) p(s) = \sum_{s \in A} p(s) 1 + \sum_{s \in \bar{A}} p(s) 0 = p(A).$$

First we reformulate Theorem 1 page 503 in Rosen as the following result about indicator random variables:

$$I_{\bigcup_{i=1}^{n} A_i} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\{I \subset \{1,2,\ldots,n\} : |I| = k\}} I_{A_I}, \tag{1}$$

where $A_I = A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}$ when $I = \{i_1, i_2, \ldots, i_k\}$

Then take expected value on both sides and use linearity of expectation:

$$E(I_{\bigcup_{i=1}^{n} A_i}) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\{I \subset \{1,2,\ldots,n\} : |I| = k\}} E(I_{A_I}), \tag{2}$$

Now use that the expected value of the indicator function I_A is $p(A)$:

$$p(\bigcup_{i=1}^{n} A_i) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\{I \subset \{1,2,\ldots,n\} : |I| = k\}} p(A_I), \tag{3}$$

$$p(\bigcup_{i=1}^{n} A_i) = \sum_{k=1}^{n} (-1)^{k-1} S_k, \tag{4}$$

We can also prove (1) by observing that the following equality holds for all $s \in S$:

$$0 = (I_A(s) - I_{A_1}(s))(I_A(s) - I_{A_2}(s)) \ldots (I_A(s) - I_{A_n}(s)), \tag{5}$$

where $A = \bigcup_{i=1}^{n} A_i$. To see this simply observe that both sides evaluate to 0 when $s \notin A$ and if $s \in A_i$ for some i then the term $(I_A(s) - I_{A_i}(s))$ is zero. Now we get (1) by expanding (5). Here we use that $I_A^k(s) = I_A(s)$ for all $k \geq 1$ and similarly for all A_i's. You should try to make this calculation to see that you really get (1).

- At the lecture on November 24 I covered the following stuff on the chromatic polynomial of a graph. Let t be a positive integer. A **t-colouring** of a graph $G = (V, E)$ is any mapping $f : V \to \{1, 2, \ldots, t\}$. Hence a graph on n vertices has $t^{|V|}$ different t colourings. We say that a t-colouring f of G is a **proper t-colouring** if $f(u) \neq f(v)$ whenever $uv \in E$, that is, uv is an edge of G.

  The **chromatic polynomial of a graph** $G = (V, E)$, denoted $P_G(t)$ is the number of proper t colourings of G. For a given graph $G = (V, E)$ and an edge $e = uv \in E$ we denote by $G - e$ the graph we obtain by deleting the edge e from G, that is, $G' = (V, E \setminus \{e\})$. We also denote by G/e the graph we obtain by **contracting** the edge e, that is, we delete the edge uv, identify u and v into one new vertex w and replace all edges in G of the kind xy where $x \in V - \{u, v\}$ and $y \in \{u, v\}$ by the edge xw.

Proposition 0.2 *For every natural number t and every graph $G = (V, E)$ we have*

$$P_G(t) = P_{G-e}(t) - P_{G/e}(t), \tag{6}$$

where e is an arbitrary edge of G.

Proof: We show that $P_{G-e}(t) = P_G(t) + P_{G/e}(t)$. This follows from the sum rule: If $e = uv$, then the set of proper colourings of $G - e$ can be divided into those that colour u and v with different colours and those that colour them by the same colour. This shows that $P_{G-e}(t) \leq P_G(t) + P_{G/e}(t)$. On the other hand, every proper t-colouring of G is clearly also a proper t-colouring of $G - e$ and given any proper t-colouring of G/e we get a proper t-colouring of $G - e$ by undoing the contraction and deleting the edge uv. Thus $P_{G-e}(t) \geq P_G(t) + P_{G/e}(t)$. ◇

Theorem 0.3 *Let G be obtained from two distinct complete graphs on 3 vertices by identifying one vertex from the first with one vertex in the second, that is, $G = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{12, 13, 23, 34, 35, 45\}$. Then $P_G(t) = t(t-1)^2(t-2)^2$.*

Proof: Let $e = 45$. Then $H = G - e$ has edges $\{12, 13, 23, 34, 35\}$ and $W = G/e$ is isomorphic to the graph vertex set $\{1, 2, 3, 4\}$ and edges $\{12, 13, 23, 34\}$. If we delete the edge 35 from H we get a graph which is W plus one extra vertex connected to nothing. Hence $P_H(t) = (t-1)P_W(t)$. To find $P_W(t)$ we consider the edge 34. Deleting 34 from W gives us the complete graph K_3 on 3 vertices plus an extra edge connected to nothing. Hence $P_W(t) = (t-1)P_{K_3}(t)$. Clearly $P_{K_3}(t) = t(t-1)(t-2)$ so $P_W(t) = t(t-1)^2(t-2)$. Inserting this above we get $P_H(t) = t(t-1)^3(t-2)$. Finally we have

$$P_G(t) = P_H(t) - P_W(t) = t(t-1)^3(t-2) - t(t-1)^2(t-2) = t(t-1)^2(t-2)^2.$$

◇

We can also prove Theorem 0.3 via the principle of inclusion-exclusion as we show below. Note that this is more cumbersome but still it illustrates the usefulness of this counting tool.

Let $e_1, e_2, \ldots, e_6$ be an ordering of the six edges of G and let the event A_i be that both endvertices of A_i get the same colour. Use N_j, $j = 1, 2, \ldots, 6$ for the number of events in which j of the events $A_1, A_2, \ldots, A_6$ occur. That is, N_1 is the number of t-colourings where precisely one of the edges has both endvertices coloured with the same colour, N_2 is the number of t-colourings where precisely two of the edges have both endvertices coloured with the same colour, etc. We wish to find the number of proper t-colourings which, according to the principle of including and excluding, is the same as the number

$$N(A'_1, A'_2, \ldots, A'_6) = N - \sum_{i=1}^{6} (-1)^i N_i. \tag{7}$$

N_1: $N_1 = 6t^4$ since the two endvertices of some edge must get the same colour (t choices) and the other 3 vertices can get any colour. We can choose the special edge in $\binom{6}{1} = 6$ ways.

N_2: $N_2 = 15t^3$ as we can choose the two special edges in $\binom{6}{2} = 15$ ways and each of these have t^3 possible colourings (if the special edges cover 3 vertices we have t choices for these and t for each of the remaining 2 vertices. If they cover 4 vertices then we have t choices for each of the two edges and t choices for the last vertex).

N_3**:** $N_3 = 18t^2 + 2t^3$. This is seen as follows: there are $\binom{6}{3} = 20$ different ways to select the 3 edges where both ends will be coloured the same. If these 3 edges form one of the 2 triangles we have t choices for these vertices and t for each of the last two vertices. On the other hand if the 3 edges cover 4 vertices all these vertices must get the same colour (t choices) and the remaining vertex also has t choices. If the 3 edges cover all vertices then there must be two connected components in the graph consisting of the 5 vertices and these 3 edges so again we get t^2 colourings.

N_4**:** $N_4 = 9t + 6t^2$. This is seen as follows: there are $\binom{6}{4} = 15$ different ways to select the 4 edges where both ends will be coloured the same. If these edges cover 4 vertices of G then it is the graph W above and there are 4 copies of this in G. These vertices get one of t colours and the remaining vertex also has t choices. If the 4 edges cover all vertices but not in a connected way, then they form a triangle and the opposite edge from the other triangle. There are 2 of these and they can also be coloured in t^2 ways. The remaining possibility is that the 4 edges form a spanning tree of G and hence there are t choices.

N_5**:** $N_5 = 6t$ since there are $\binom{6}{5} = 6$ ways of choosing the 5 edges where both ends will be coloured the same. They always give a connected spanning subgraph of G.

N_6**:** $N_6 = t$ this is clear.

Inserting in (7) we get

$$\begin{aligned} N(A_1', A_2', \ldots, A_6') &= t^5 - 6t^4 + 15t^3 - [18t^2 + 2t^3] + [9t + 6t^2] - 6t + t \\ &= t^5 - 6t^4 + 13t^3 - 12t^2 + 4t \\ &= t(t-1)^2(t-2)^2. \end{aligned}$$