Exam 2010 Prb 3

suppose we n_1 correct and n_2 wrong

then score $s = n_1 + (-n_2) = n_1 - n_2$

so with n questions and n_1 correct

the score is $n_1 - (n - n_1) = 2n_1 - n$

Students

A correct with prob $\frac{3}{5}$

B - - - - — $\frac{4}{5}$

C - - - - - - $\frac{1}{2}$

X_A, X_B, X_C denote the score for each of A, B, C

a) Find $E(X_A)$, $E(X_B)$ and $E(X_C)$

$$E(X_A) = 10 \cdot \frac{3}{5} - 10 \cdot \frac{2}{5} = 2$$

$$\left(10 \cdot \left(\frac{3}{5} - \frac{2}{5}\right)\right)$$

$$E(X_B) = 10 \cdot \left(\frac{4}{5} - \frac{1}{5}\right) = 6$$

$$E(X_C) = 10 \cdot \left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

b) $P(X_c > 0)$?

This is the same as answering correctly on at least 6 questions

also $P(X_c > 0) = P(X_c < 0)$

$$P(X_c \geq k) = \sum_{j=k}^{10} \binom{10}{j} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)^{10-j}$$

$$P(X_c) = \frac{1 - P(X_c = 0)}{2}$$

$$P(X_c = 0) = \binom{10}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5 = \frac{63}{256}$$

$$P(X_c) = \frac{1 - \frac{63}{256}}{2} = \frac{193}{512}$$

Similar calculations (using binomial formula) for B and A

e.g. $P(X_B > 0) = \sum_{j=6}^{10} \binom{10}{j} \left(\frac{4}{5}\right)^j \left(\frac{1}{5}\right)^j = 0.9672$

d) Suppose 8 students behave like C
Find expected no of those 8 students
who will pass (score ≥ 2)

answer $E(\#\text{passing}) = 8 \cdot \frac{193}{512} \sim 3$

Suppose we increase # questions to
$n = 100, 200$ or 500 and passing $\Leftrightarrow \geq 2$ points

$$p = P\left(X_C \geq (1+0.2)\cdot\mu\right) \leq \left[\frac{e^{0.2}}{1.2^{1.2}}\right]^{\mu}$$

This gives

$n = 100$ $p \leq 0.3909$

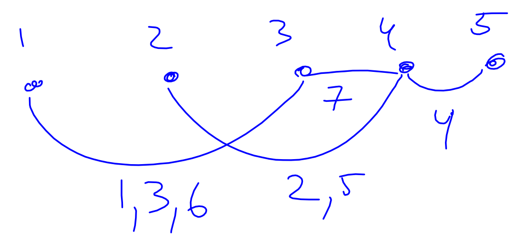
200 $p \leq 0.1528$

500 $p \leq 0.0091$

Problem 5

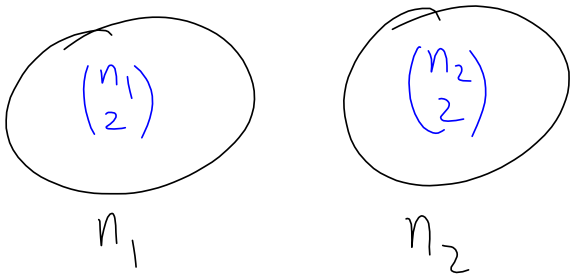
Build a connected graph by adding random edges starting from no edges

example



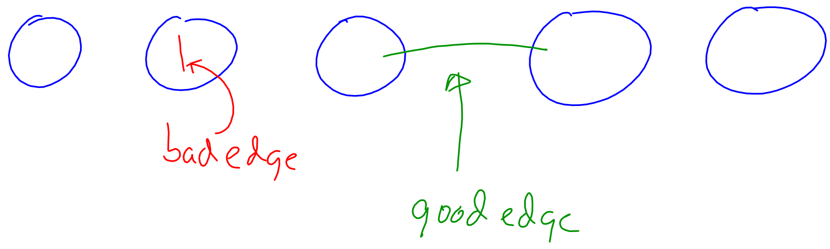
How many edges could be added before we have a connected graph?

$\underline{\infty}$ and even if no two edges are the same



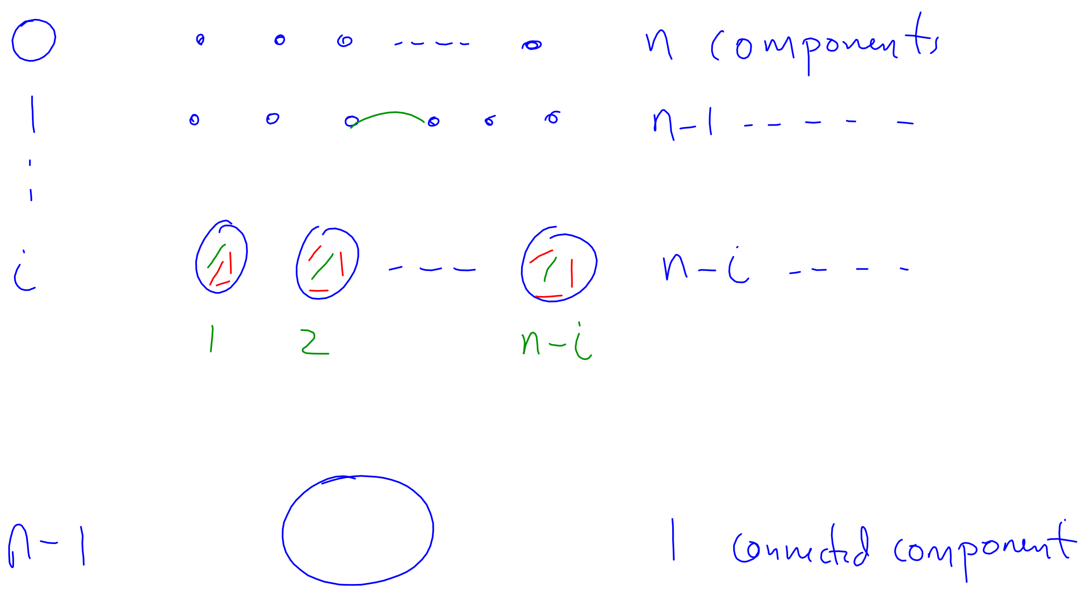
$n = n_1 + n_2$

$\sim \frac{n^2}{4}$ edges if $n_1 = n_2 = \frac{n}{2}$



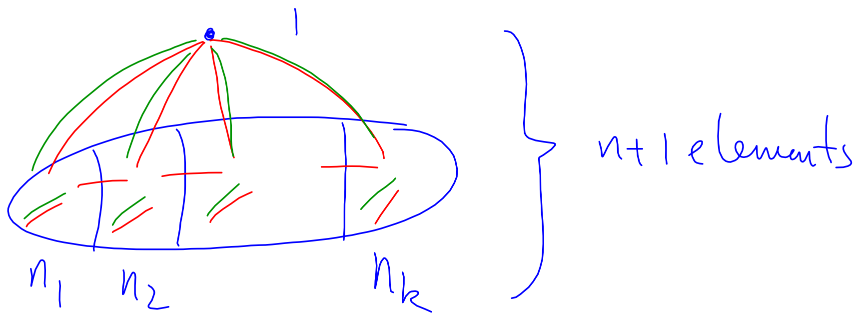
need exactly $n-1$ good edges
so define phases w.r.t # good edges
collected so far

phase



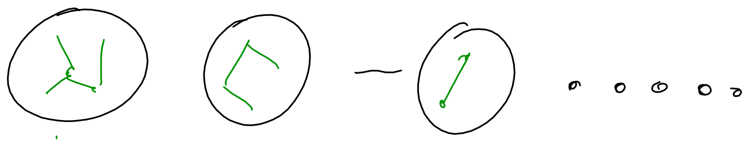
a) claim $\binom{n+1}{2} > \binom{n_1+1}{2} + \binom{n_2+1}{2} + \cdots + \binom{n_k+1}{2}$

when $n = \sum_{i=1}^{k} n_i \quad k > 1$



c) How many edges do we expect to generate in phase i?

phase i $\Leftrightarrow$ i good edges and some k non-trivial components containing these



i_1 $\qquad$ i_2 $\qquad$ i_k $\leftarrow$ # of good edges

$n_1 = i_1 + 1$ $\qquad$ $n_2 = i_2 + 1$ $\qquad$ $n_k = i_k + 1$

$$i = \sum_{j=1}^{k} i_j$$

The next edge is good unless it is inside one of the k non-trivial components.

no such edges is $\binom{n_1}{2} + \binom{n_2}{2} + \dots \binom{n_k}{2}$

$$= \binom{i_1+1}{2} + \binom{i_2+1}{2} + \dots + \binom{i_k+1}{2} \underset{(a)}{<} \binom{i+1}{2}$$

This means that in phase i we have at least $\binom{n}{2} - \binom{i+1}{2}$ choices for a new good edge

so $p = \text{prob}(\text{next edge good}) \geq \dfrac{\binom{n}{2} - \binom{i+1}{2}}{\binom{n}{2}}$

Hence if $X_i = \#$edges in phase i we have

$$E(X_i) = \frac{1}{p} \leq \frac{\binom{n}{2}}{\binom{n}{2} - \binom{i+1}{2}}$$

$X =: X_0 + X_1 + \dots + X_{n-2}$ total no of edges

$$E(X) = \sum_{i=0}^{n-2} E(X_i)$$

$$\leq \sum_{i=0}^{n-2} \frac{\binom{n}{2}}{\binom{n}{2} - \binom{i+1}{2}} = \binom{n}{2} \sum_{i=0}^{n-2} \frac{1}{\binom{n}{2} - \binom{i+1}{2}}$$

In f) you show $\frac{n-1}{2}\left(\frac{1}{\binom{n}{2} - \binom{i+1}{2}}\right) \leq \frac{1}{n-i}$

so $E(X) \leq n \sum_{i=0}^{n-2} \frac{1}{n-i} = \Theta(n \log n)$

Kleinberg-Tardòs Prb 7

General SAT (not just 3 literals per clause)

e.g. $(X_1) \wedge (\bar{X}_1) \wedge (X_2 \vee \bar{X}_3) \wedge \cdots$
$(X_1 \vee \bar{X}_3 \vee X_7 \vee \bar{X}_{11} \cdots \vee X_{222})$

a) As for 3-SAT choose random truth ass. and

let $X_i = \begin{cases} 1 & \text{if } C_i \text{ satisfied} \\ 0 & \text{else} \end{cases}$

$$P(X_i = 1) = \left(1 - \left(\frac{1}{2}\right)^{|C_i|}\right) \geq \frac{1}{2}$$

So $E(\sum X_i) \geq \frac{1}{2}$

So $\exists$ truth ass s.t $\sum X_i \geq \frac{1}{2} k$ $k = \#$ clauses

$(x_1) \wedge (\bar{x}_1) \wedge (x_2) \wedge (\bar{x}_2) \wedge \cdots \wedge (x_r) \wedge (\bar{x}_r)$

precisely half will be satisfied.

b) assume no pair of clauses

(X_i) and $(\bar{X}_i)$

prove that now $\exists$ a truth assignment satisfying at least $\frac{3}{5}$ of all clauses

Let $r = \#$ clauses of size 1

case 1 $r \geq \frac{3}{5}k$ $k = \#$ clauses

then just all of these true

case 2 $r < \frac{3}{5}k$

Consider a random truth ass.

and calculate expected no of satisfied cl.

There are $k-r$ clauses of size ≥ 2

and expect at least $\frac{3}{4}$ of these satisfied

and $\frac{r}{2}$ of the singletons to be satisfied

$$E(X) \geq \frac{r}{2} + \frac{3}{4}(k-r) = \frac{r}{2} + \frac{3}{4}(k-r)$$

$$= \frac{3k}{4} - \frac{1}{4}r$$

$$\geq \frac{3k}{4} - \frac{3}{20}k = \frac{6}{10}k$$