Cormen 5.1 Hirering problem

Interview n candidates each with a "value" v_i $i=1,2 \ldots n$ interviewed in order $1,2 \ldots n$

Rule: If v_i is better than all previous
then hire v_i and fire current candidate
(unless $i=1$)

Randomized version: randomize candidates and run hirenng alg.

Q: what is the expected # of hirenngs?

Analysis

For $i=1,2 \ldots n$ let $X_i = \begin{cases} 1 & \text{if } i\text{th cand is hired} \\ 0 & \end{cases}$

Then $X = \sum X_i$ is # persons hired

$P(X_i = 1) = \frac{1}{i}$ since v_i must be better than $v_1, \ldots, v_{i-1}$

$$E(X) = E\left(\sum_i X_i\right) = \sum_i E(X_i) = \sum_i \frac{1}{i} = H(n)$$

$H(n) = \Theta(\ln n)$

5.3 Generating random permutations

Def a permutation of n elements is random if it has probability $\frac{1}{n!}$ of being chosen from the set of all the $n!$ perms.

Alg 1
1. assign each element a (i) (n elements) random integer q_i from $[1, n^3]$
2. If elements a and b get same number then repeat 1.
3. Sort elements according their numbers such $p_{i_1} < p_{i_2} < \cdots < p_{i_n}$
4. return permutation (order) $i_1, i_2, \ldots, i_n$

probability that all p_i's are distinct is at least $\left(1 - \frac{1}{n}\right)$

Have to show that the permutation returned is a random one.

Show only that $\pi = 123\cdots n$ has prob $\frac{1}{n!}$

E_i: element i gets position i in π

we want $p(E_1 \cap E_2 \cap \cdots \cap E_n)$

$$p(E_1) = \frac{1}{n}$$

$$p(E_2 | E_1) = \frac{1}{n-1}$$

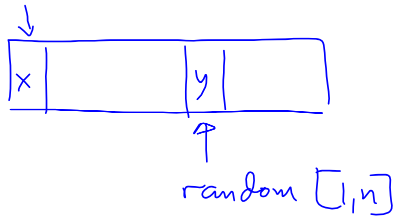
$$\vdots$$

$$p(E_i | E_1 \cap \cdots \cap E_{i-1}) = \frac{1}{n-i+1}$$

$$p(E_1 \cap E_2 \cap \cdots \cap E_n) = p(E_1)p(E_2|E_1) \cdots p(E_n|E_1 \cap \cdots \cap E_{n-1})$$
$$= \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{1} = \frac{1}{n!}$$

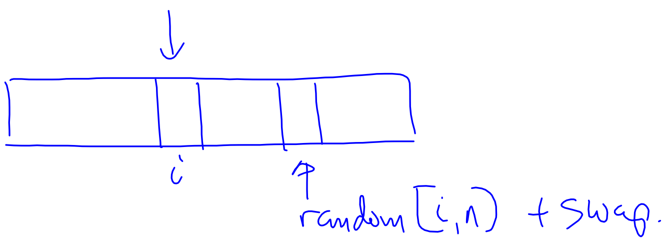
Algorithm 2

step 1



swap x, y

Step i



Q: Does this generate each perm with the same probability $\left(\frac{1}{n!}\right)$?

k-permutation of n-set S: an ordered subset on k element of S

k-permutations $\dfrac{n!}{(n-k)!}$

Want to prove:

(I): after step k every k-perm of S has probability $\dfrac{(n-k)!}{n!}$ of occupying positions 1, 2 .. k

(I) holds after step 1

since each element has prob $\frac{1}{n}$ of being in position 1 assume (I) holds after step $i-1$

show that it holds after step i

Consider a i-perm. $\langle x_1 x_2 \dots x_i \rangle$ of S

E_1: positions $1, 2 \dots i-1$ are occupied by $x_1 x_2 \dots x_{i-1}$

E_2 position i is occupied by x_i

$$p(E_1 \cap E_2) = p(E_2 | E_1) p(E_1)$$

$$= \frac{1}{n-i+1} \cdot \frac{(n-i+1)!}{n!} = \frac{(n-i)!}{n!}$$

so (I) holds ✓

5.4 Cormen (read **5.4.1 and 5.4.2**)

5.4.3 Estimating length of streaks in n coin flips

streak : subsequence of identical out comes e.g.

$\underbrace{hh}\underbrace{t}\underbrace{h}\underbrace{ttt}\underbrace{hhh}$

k-streak : streak of k elements

Define event A_{ik} as

A_{ik} : only heads in each of the flips $i, i+1, \ldots, i+k-1$

$p(A_{ik}) = 2^{-k}$ so with $k = 2\lceil \log_2 n \rceil$

we have

$p(A_{i\,2\lceil \log n \rceil}) = 2^{-2\lceil \log n \rceil} \leq 2^{-2\log n} = (2^{\log n})^{-2} = \frac{1}{n^2}$