Ex 4 Derangements

a derangement of $\{1, 2, \dots, n\}$ is permutation $\pi \to \{1,2,\dots,n\} \to \{1,2,\dots n\}$ s.t $\pi(i) \neq i$ $\forall i$

Thm # derangements (D_n) of an n-set is $D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}\right]$

P: π is a derangement if and only if it has none of the properties $P_1, P_2 \dots P_n$, when $P_i : \pi(i) = i$

$1,2 \dots i \dots n \xrightarrow{\pi} \sim i \sim$

$N(P_i) = (n-1)!$ $\quad i = 1, 2 \dots n$ $\quad \binom{n}{1}$

$1,2 \dots i \dots j \dots n \to \dots i \dots j \dots$

$N(P_i P_j) = (n-2)!$ $\quad 1 \leq i < j \leq n$ $\quad \binom{n}{2}$

$N(P_{i_1} P_{i_2} \dots P_{i_k}) = (n-k)!$ $\quad 1 \leq i_1 < i_2 \dots < i_k \leq n$ $\quad \binom{n}{k}$

$D_n = n! -$ # of sols with at least one fixpoint

$= n! - \left[\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! \dots + (-1)^{n+1}\binom{n}{n}(n-n)!\right]$

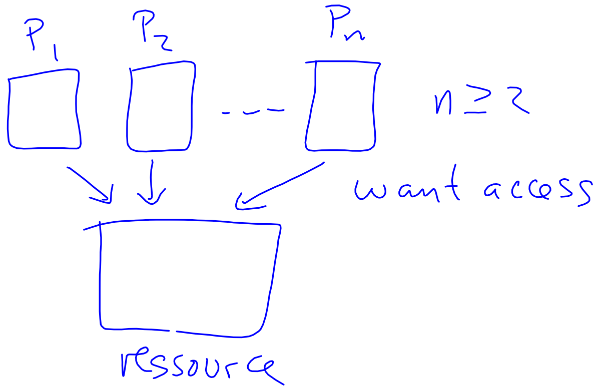
$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \dots + (-1)^n \binom{n}{n}(n-n)!$

$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1) \frac{1}{n!}\right]$

$\sim \frac{n!}{e}$

switch to notes (Chap. 13, Kleinberg & Tardós)

Section 13.1



Rule: If, at any given times, precisely one processor asks for access, then this is granted. otherwise no process gets access

Use the following randomized strategy (for a fixed p, known to all):

In each round, each P_i attempts access with probability p.

Observations

1. If $p=1$ nobody ever gets access
2. If $p=0$ -- -- -- -- -- -

Goal find a good value for p!

Events A_{it}: process i attempts access in round t

$P(A_{it}) = p$, $P(\overline{A_{it}}) = 1-p$ by protocol

S_{it}: P_i gets access in round t

so $S_{it} = A_{it} \cap \bigcap_{j \neq i} \overline{A_{jt}}$

$p(S_{it}) = p(1-p)^{n-1} = f(p)$

wish to find p s.t $f(p)$ is maximized

$$f(p) = p(1-p)^{n-1}$$

$$f'(p) = (1-p)^{n-1} - p(n-1)(1-p)^{n-2}$$

$$= (1-np)(1-p)^{n-2}$$

So when $0<p<1$ we get $f'(p)=0$ precisely when $p=\frac{1}{n}$ and this is a maximum for $f(p)$

$$f\left(\frac{1}{n}\right) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1} = p(S_{i,t})$$

Calculus

(a) $\left(1-\frac{1}{n}\right)^{n}$ $\quad \frac{1}{4} \nearrow \frac{1}{e}$ for $n \to \infty$, $n \geq 2$

(b) $\left(1-\frac{1}{n}\right)^{n-1}$ $\quad \frac{1}{2} \searrow \frac{1}{e}$ for $n \to \infty$, $n \geq 2$

By (b) $\frac{1}{en} \underset{(\square)}{\leq} p(S_{it}) \leq \frac{1}{2n}$ $\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}$

F_{it}: P_i has no success in any of the rounds $1, 2, \ldots t$

$$F_{it} = \bigcap_{j=1}^{t} \overline{S_{ij}}$$

$$p(F_{it}) = \prod_{j=1}^{t} p(\overline{S_{ij}}) = \left(1 - \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}\right)^t$$

$$\leq \left(1 - \frac{1}{en}\right)^t \text{ by } (\square)$$

recall that $\left(1-\frac{1}{x}\right)^x \to \frac{1}{e}$ as $x \to \infty$

So taking $t = \lceil en \rceil$ then

$$p(F_{it}) \leq \left(1 - \frac{1}{en}\right)^{\lceil en \rceil} \leq \left(1-\frac{1}{en}\right)^{en} \leq \frac{1}{e} \text{ by above}$$

and if we take $t = \lceil en \rceil \cdot c \cdot n$ then

$$p(F_{it}) \leq \left(1-\frac{1}{en}\right)^{\lceil en \rceil \cdot c \ln n}$$

$$= \left(\left(1-\frac{1}{en}\right)^{\lceil en \rceil}\right)^{c \ln n} \leq \left(\frac{1}{e}\right)^{c \ln n}$$

$$= \left((e^{-1})^{\ln n}\right)^c = n^{-c} = \frac{1}{n^c}$$

in particular with $c = 2$:

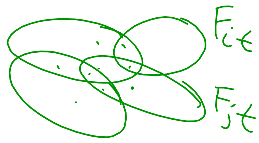
$$p(F_{it}) \leq \frac{1}{n^2}$$

Q: How rounds before all P_i's had access at least once?

Know: $P(F_{it}) \leq \frac{1}{n^2}$ with $t = 2 \ln n \lceil en \rceil$

We look for $P(F_{1t} \cup F_{2t} \cup \ldots \cup F_{nt})$ (this is the prob that some process never access)

$$P\left(\bigcup_{i=1}^{n} F_{it}\right) \leq \sum_{i=1}^{n} P(F_{it}) \quad \text{(Union bound)}$$



so $P\left(\bigcup_{i=1}^{n} F_{it}\right) \leq \sum_{i=1}^{n} \frac{1}{n^2} = \frac{1}{n}$

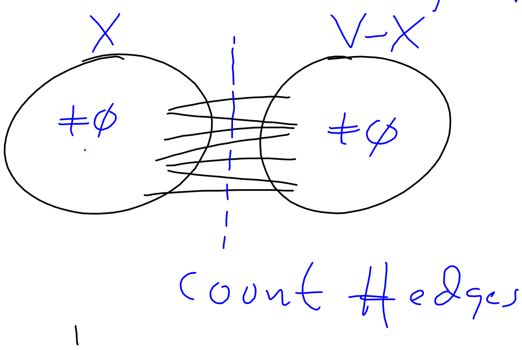
when $t = 2 \ln n \lceil en \rceil$

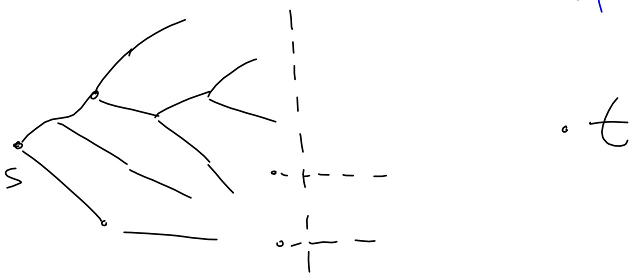
if t was $4 \ln n \lceil en \rceil$ then we would get

$$P\left(\bigcup_{i=1}^{n} F_{it}\right) \leq \sum_{i=1}^{n} \frac{1}{n^4} = \frac{1}{n^3}$$

13.2 Min cuts in graphs

$G=(V,E)$





We wish to find a cut $(X, V-X)$ s.t # of edges between X and $V-X$ is minimized.

Not good to try all choices for X: because there are $2^n - 2$ such $n = |V|$

There exist a $O(n^3)$ deterministic alg

Our goal is to make a simpler randomized algorithm.