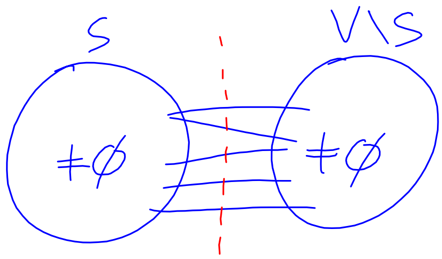
Minimum Cuts in Graphs



size of cut = # edges
(here 5)

Min cut problem

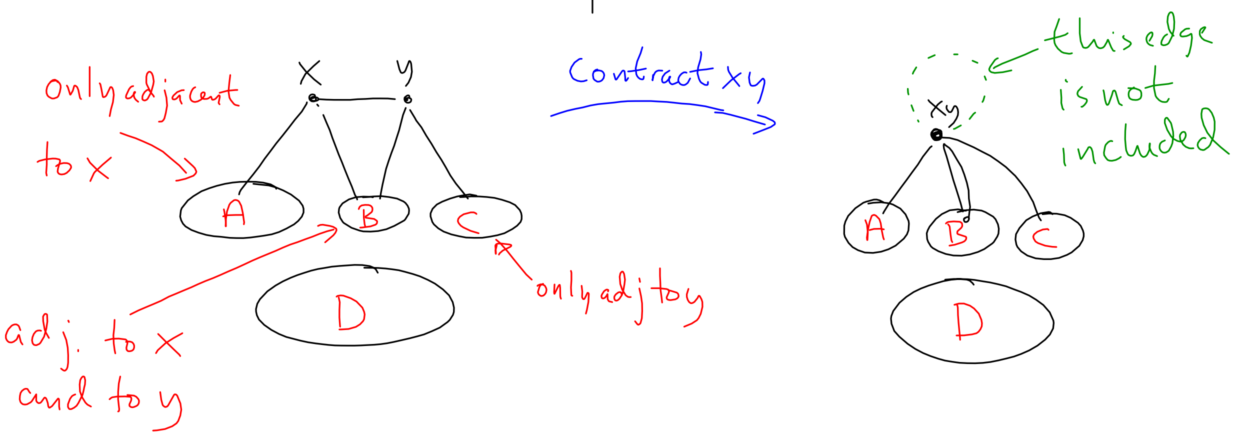
Input a graph $G=(V,E)$

Output: a minimum cut $(S, V\setminus S)$ in G, that is a partion $(S, V\setminus S)$ which mimizes # edges accross

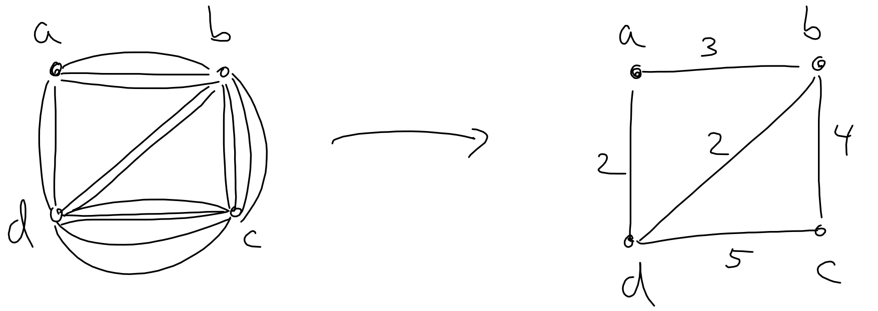
assumptions / definitions

G may have multiple edges 

We are using contraction.



How do we ~~cope~~ represent with parallel edges



representation of G

Example of a contraction

G $\qquad\qquad$ G\be

Karger's algorithm

On input $G=(V,E)$ $n=|V|$ (assume that G is connected)

Repeat $n-2$ times

pick a random edge $e=uv$

Contract uv $(G \leftarrow G/uv)$

return the cut $(S, V\setminus S)$

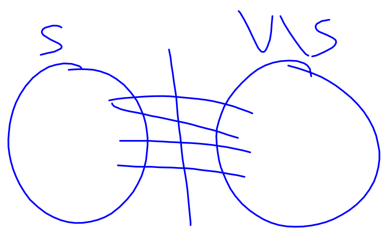
where $S=\{v \mid v \text{ contracted into } v_1\}$

$V\setminus S=\{v \mid v \text{ contracted into } v_2\}$

and $V(G)=\{v_1,v_2\}$ at the end of loop

$v_1 \overset{r}{\text{———}} v_2$ ← in final G

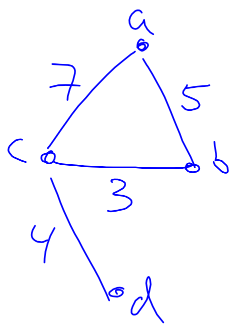
corresponds to

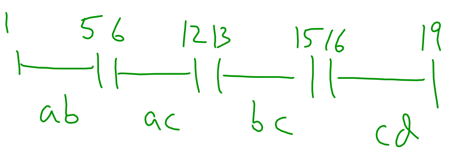


$\Sigma = r$

What does it mean to select a random edge in the current $G'=(V',E')$?

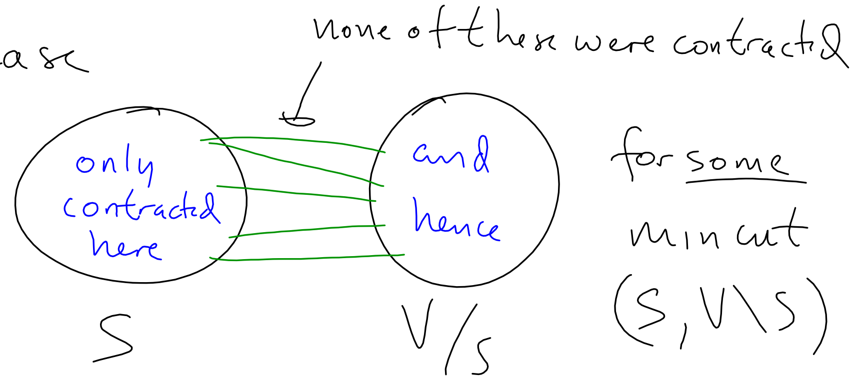
Ex $G'=$





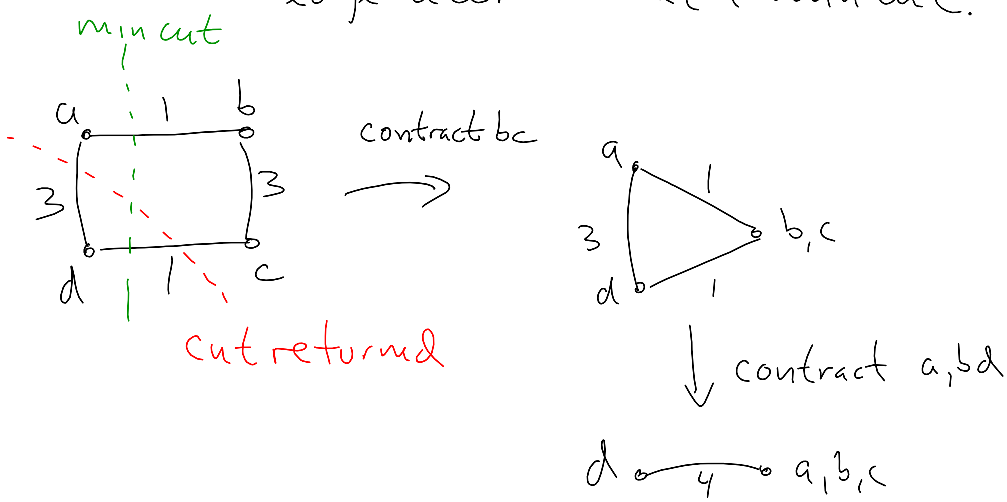
How good is the algorithm?

Good case



for some min cut $(S, V \setminus S)$

Bad case: contracted at least one edge accross each min cut.



Theorem For a fixed min cut $(S, V\setminus S)$ the probability that this is the cut returned by the algorithm is at least $\frac{1}{\binom{n}{2}} = \frac{2}{n(n-1)}$

Consider K repetitions of the algorithm: the probability that we do not return $(S, V\setminus S)$ in any run is at most $q \leq \left(1-\frac{2}{n(n-1)}\right)^K$

recall $\left(1-\frac{1}{x}\right)^x \leq \frac{1}{e}$

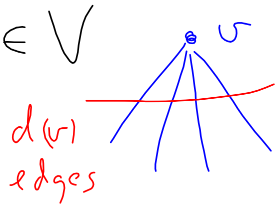
So with $K = n(n-1)\ln n$

$$q \leq \left[\left(1-\frac{2}{n(n-1)}\right)^{n(n-1)}\right]^{\ln n} \leq \frac{1}{e}^{2\ln n} = n^{-2} = \frac{1}{n^2}$$

proof of theorem

Fix a min cut $(A, V \setminus A)$ and let k be the size of this (and all) min cut

Since every min cut has at least k edges we have that $d(v) \geq k \quad \forall v \in V$



Then $E = \dfrac{\sum d(v)}{2} \geq \dfrac{nk}{2}$

E_1: our cut survives first contraction

$p(E_1) = 1 - p(\bar{E}_1)$

and $p(\bar{E}_1) = \dfrac{\#\text{edges in our cut}}{|E|} \leq \dfrac{k}{\frac{nk}{2}} = \dfrac{2}{n}$

So $p(E_1) \geq \left(1 - \frac{2}{n}\right)$

Define E_j similarly so

E_j: our cut not killed by jth contraction

Suppose all of $E_1, E_2, \ldots, E_j$ occour

(= cut still alive)

Then the graph G_j (after j contractions)

has n-j vertices and $E(G_j) \geq \frac{(n-j)k}{2}$ edges

since $d_{G_j}(v) \geq k \;\; \forall v \in V(G_j)$

So as before our cut survives the (j+1)th

Contraction with probability at least

$$1 - \frac{k}{\frac{(n-j)k}{2}} = 1 - \frac{2}{n-j}$$

So we have

$$p(E_1) \geq \left(1 - \frac{2}{n}\right)$$

$$p(E_{j+1} \mid E_1 \cap E_2 \cap \ldots \cap E_j) \geq \left(1 - \frac{2}{n-j}\right)$$

So we get $p(E_1 \cap E_2 \cap \ldots \cap E_{n-2})$

$$= p(E_1)\, p(E_2 \mid E_1) \cdot p(E_3 \mid E_1 \cap E_2) \cdots p(E_{n-2} \mid E_1 \cap \ldots \cap E_{n-3})$$

$$\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right) \cdots \left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{3}\right)$$

$$= \frac{\cancel{n-2}}{n} \cdot \frac{\cancel{n-3}}{n-1} \cdot \frac{n-4}{\cancel{n-2}} \cdots \frac{\cancel{3}}{\cancel{5}} \frac{2}{4} \frac{1}{\cancel{3}} = \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}$$

<u>Corollary</u> Every connected graph G on n vertices has at most $\binom{n}{2}$ distinct min cuts.

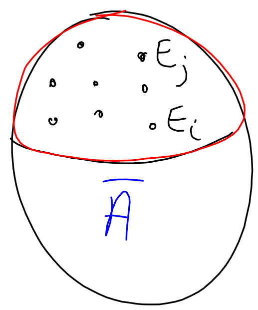
<u>P:</u> let $(A_1, V\setminus A_1), (A_2, V\setminus A_2) \ldots\ldots (A_L, V\setminus A_L)$ be the set of distinct min cuts.

We saw that Karger's alg returns any fixed min cut $(A_i, V\setminus A_i)$ with prob at least $\frac{1}{\binom{n}{2}}$

The event $\overset{E_i}{\vee}$ that the alg. returns $(A_i, V\setminus A_i)$ is <u>disjoint</u> from the event E_j that it returns $(A_j, V\setminus A_j)$ only one cut returned

<u>so</u> $p(E_1 \cup E_2 \cup \ldots \cup E_L)$ (the prob. that a min cut is returned) is at least $\sum_{j=1}^{L} \frac{1}{\binom{n}{2}} = \frac{L}{\binom{n}{2}}$

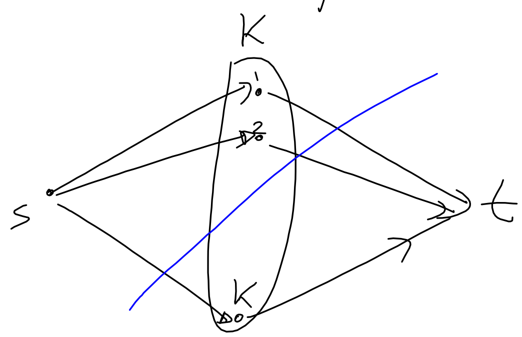
$A = E_1 \cup \ldots \cup E_L$



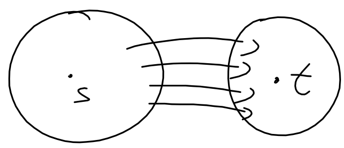
We also have $1 \geq p(E_1 \cup \ldots \cup E_L)$

so $1 \geq \frac{L}{\binom{n}{2}} \Rightarrow L \leq \binom{n}{2}$

For directed graphs there no such bound:



Look for (s,t)-cut



2^k (s,t)-cuts of size k

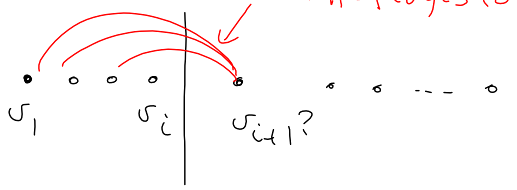
and that is the minimum.

Max back orderings of a graph $G=(V,E)$
(mbo)

Ordering $v_1, v_2, \dots v_n$ of V

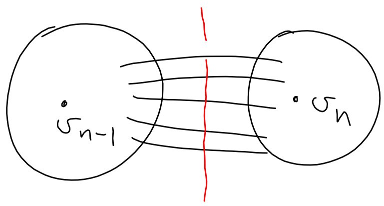
s.t $\forall\, i \in 2, \dots n-1$:

max # of edges to $V_i = \{v_1, v_2, \dots v_i\}$



among all $v \in V \setminus V_i$

<u>Theorem</u> If $v_1, v_2 \dots v_n$ is a mbo of G
then $\lambda(v_{n-1}, v_n) = d(v_n)$

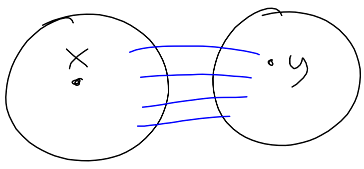


edges $\geq d(v_n) = \lambda(v_{n-1}, v_n)$

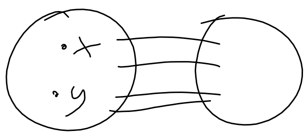
Note that if $\lambda(G)$ denotes valm of a min cut in G and x, y are distinct vertices

then $\lambda(G) = \min\{\lambda(G \setminus \{x,y\}), \lambda(x,y)\}$

Two types of min cuts



at least $\lambda(x,y)$ edges



at least $\lambda(G \setminus \{x,y\})$ edges

So if we run $n-2$ mbo's
record $\lambda(v_{n-1}, v_n)$
Contract v_{n-1}, v_n and repeat
We can find a min cut.