Ex 5 # of solutions to $x_1+x_2+x_3=11$

$x_i \geq 0$

answer same as # ways we can choose 11 places for '$*$''s out of 13:

$$\overset{x_1=4}{****} \mid \overset{x_2=5}{*****} \mid \overset{x_3=2}{**}$$

$$x_1 \qquad x_2 \qquad x_3$$

$$\binom{n}{k} = \binom{n}{n-k}$$

this is $\binom{13}{11} = \binom{11+(3-1)}{11} = \binom{11+(3-1)}{3-1}$

what if we insist that

$$\underset{x_1'=3}{\overset{x_1=7}{x_1 \geq 4}}, \quad \underset{x_2'=0}{\overset{x_2=2}{x_2 \geq 2}}, \quad \underset{x_3'=2}{\overset{x_3=2}{x_3 \geq 0}} \quad \text{in } x_1+x_2+x_3=11$$

So we have fixed $4+2=6$ of the 11

So 5 remains

so we need to solve

$$x_1'+x_2'+x_3' = 5 \quad (=11-6)$$

$$\binom{5+(3-1)}{5} = \binom{7}{5}$$

Permutations when some objects are equal (indistinguishable)

SUCCESS 3 S
2 C

SSUECSC

distinct permutations:

$\binom{7}{3}$ places for 'S''s

$\binom{4}{2}$ places for 'C''s

$\binom{2}{1}$ places for U then E is fixed

$$\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = \frac{7!}{3!\,2!\,1!}$$

$$\frac{7!}{\cancel{4!}\,3!}\cdot\frac{\cancel{4!}}{\cancel{2!}\,2!}\,\frac{\cancel{2!}}{1!\,1!}$$

<u>Thm</u> # permutations of n objects of k types s.t there are n_i obj. of type i $(n_1+n_2+\cdots+n_k=n)$

is $$\frac{n!}{n_1!\cdot n_2!\cdot\ldots\cdot n_k!}$$

<u>proof 1</u> n_1 places for type 1 can be chosen in $\binom{n}{n_1}$ ways

n_2 - - - — " - - — $\binom{n-n_1}{n_2}$ ways

⋮

n_{k-1} - - - - $\binom{n_{k-1}+n_k}{n_{k-1}}$

n_k - - - - one way

$$\frac{n!}{n_1!\,\cancel{(n-n_1)!}}\cdot\frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}}\cdots\frac{\cancel{(n_{k-1}+n_k)!}}{n_{k-1}!\,n_k!}$$

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

<u>proof2</u> look at permutations this way

| n_1 elements | n_2 elements | — — — | n_k elements |
type 1   type 2   type k

$n!$ perm. in total

$n_1!$ ways of writing the same elements of type 1

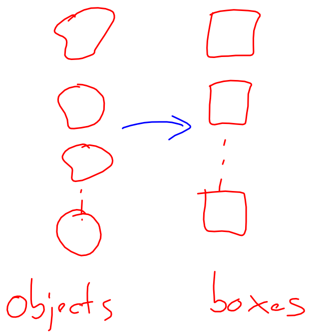
$n_2!$ — — — — — — — — — — 2

⋮

$n_k!$. — — — — — — — — — — . k

so # different is $\dfrac{n!}{n_1!\,n_2!\cdots n_k!}$

Distributing objects into boxes



$\neq, =$ $\qquad$ $\neq, =$

4 different types

distinct objects, distinct boxes

— " —, identical

identical obj, — " —

— " —, distinct

$(\neq, \neq)$

\# ways to distribute n objects into k distinct boxes such that we put n_i obj. in box i $(n_1+n_2+\cdots+n_k=n)$

We can choose the n_1 elements for box 1 in $\binom{n}{n_1}$

after this we have $n-n_1$ objects left

$\binom{n-n_1}{n_2}$ choices for objects to box 2

$\vdots$

$\binom{n-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}}$ for box $k-1$

last ones are fixed to box k

So # ways is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdot \cdots \binom{n-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}}\binom{n_k}{n_k}$$

$$= \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Ex 8 # of ways to deal 4 poker hands

is $$\frac{52!}{5!\,5!\,5!\,5!\,32!}$$

$(=, \neq)$ objects indistinguishable
boxes can be distinguished

Claim there are $\binom{n+k-1}{n}$ ways of

distributing n identical obj into k dist. bxs

$* * * \mid * * \mid * \; ---- \mid * \mid$

box1 box2 box $k-1$ box k

$(\neq, =)$

Ex 10 4 persons $A, B, C, D \longrightarrow$ 3 identical offices

1 . 4 in one office ABCD, ,

$\binom{4}{3}$. 3 in one office, 1 in another (ABC, D) (ABD, C) (ACD, B) (BCD, A)

$\binom{4}{2}/2$. 2 in one and 2 in another (AB, CD) (AC, BD) (AD, BC)

6 . 2, 1 and 1 (AB, C, D), (AC, B, D) (AD, B, C) (BC, A, D), (BD, A, C) (CD, A, B)

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alternative way of counting: # offices used

1. office 1 possibility 1

2. offices (3,1) or (2,2) : $\binom{4}{3} + \frac{\binom{4}{2}}{2} = 4 + 3 = 7$

3. offices (2,1,1) $\binom{4}{2} = 6$

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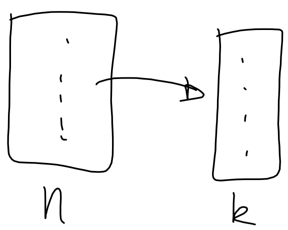
related to # onto mappings

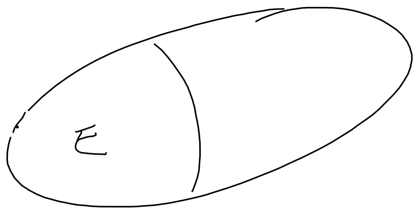


$(=,=)$

Ex 6 identical balls into 4 boxes (identical)

(6) (5,1) (4,2) (4,1,1) (3,3) (3,2,1)
(3,1,1,1) (2,2,2) (2,2,1,1)

Probability



S

sample space

ex 2 dices thrown.

Event sum=7

(1,1) (1,2) .. (1,6)

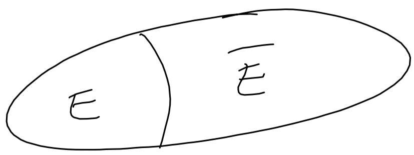
⋮

(6,1) ---- (6,6)

def $p(E) = \frac{|E|}{|S|}$

$E = \{(1,6),(2,5)(3,4),(4,3)(5,2),(6,1)\}$

$|E| = 6$, $|S| = 36$ $p(E) = \frac{6}{36} = \frac{1}{6}$

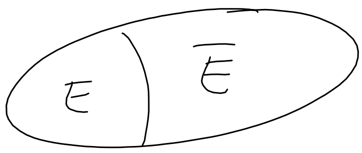


S

E our event $\bar{E}$ complementary event

(when E does not occur)

<u>observation</u> $p(\bar{E}) = 1 - p(E)$



$p(E) + p(\bar{E}) = 1$

Example of using this

prob that a random string of n bits has at least one '1'

100110

E : our random bit string has ≥ 1 '1'

1 $\bar{E}$: no '1's so only '0's

$p(E) = 1 - p(\bar{E}) = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$