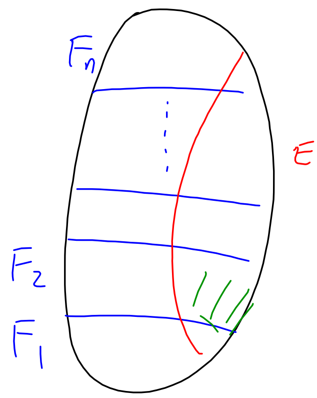
General Bayes thm.

by def
$p(A|B) = \dfrac{p(A \cap B)}{p(B)}$

$$p(F_2|E) = \frac{p(F_2 \cap E)}{p(E)} = \frac{p(E|F_2)\,p(F_2)}{p(E)}$$



$$p(E) = \sum_{i=1}^{n} p(E|F_i)\,p(F_i)$$

So $$p(F_2|E) = \frac{p(E|F_2)\,p(F_2)}{\sum_{i=1}^{n} p(E|F_i)\,p(F_i)}$$

Ex 2 sect 7.3

look at a rare disease

F : person has disease $\quad p(F) = 10^{-5}$

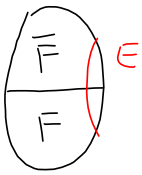
E : positive test

suppose we know the following about the test :

$$p(E|F) = \frac{99}{100}$$

$$p(\bar{E}|\bar{F}) = \frac{995}{1000}$$

We want to know $p(F|E)$ that is, the prob of having the disease if the test is positive.

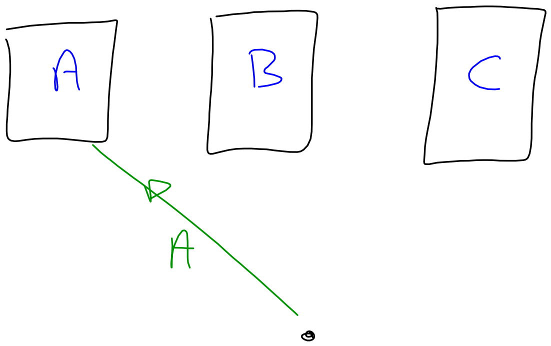


$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)\cdot p(F) + p(E|\bar{F})p(\bar{F})}$$

$$= \frac{\frac{99}{100}\cdot 10^{-5}}{\frac{99}{100}\cdot 10^{-5} + \frac{5}{1000}\left(1-10^{-5}\right)}$$

$$\sim 0{,}002.$$

Monty Hall puzzle



1.

2. host opens door B : no price

3. ? change ?

Intuition : we should change to C

Initially we have $p(E_A) = p(E_B) = p(E_C) = \frac{1}{3}$

where E_A : price behind door A

so $p(E_B \cup E_C) = \frac{2}{3}$

We learned that price is not behind door B since host opened door B

Thus $p(E_C) = \frac{2}{3}$ (as now $p(E_B) = 0$)

analysis using Bayes' formula

B_{opened} (B_o) : the host opens door B

E_A : price behind door A

E_B, E_C similar

we seek $p(E_A|B_o)$ (and $p(E_C|B_o)$)

By Bayes's formula:

$$p(E_A|B_o) = \frac{p(B_o|E_A)p(E_A)}{p(B_o|E_A)p(E_A)+p(B_o|E_B)p(E_B) + p(B_o|E_C)p(E_C)}$$

$$p(B_o|E_A) = \frac{1}{2}$$

$$p(B_o|E_B) = 0 \quad p(B_o|E_C) = 1$$

$$p(E_A) = p(E_B) = p(E_C) = \frac{1}{3}$$

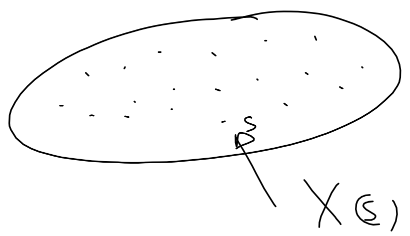
so $p(B_o) = \frac{1}{2}\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = \frac{1}{2}$

Now $p(E_A|B_o) = \dfrac{p(B_o|E_A)p(E_A)}{p(B_o)}$

$$= \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

$$p(E_C|B_o) = \frac{1\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \;!!$$

Back random variables



S samplespace

$X : S \to \mathbb{R}$

ex 2 dice $(i,j) \in \{(1,1) \cdots\cdots (6,6)\}$

$X(i,j) = i+j$

Expected value of random var X

$$E(X) \overset{\text{def}}{=} \sum_{s \in S} X(s) \cdot p(s)$$

ex one die

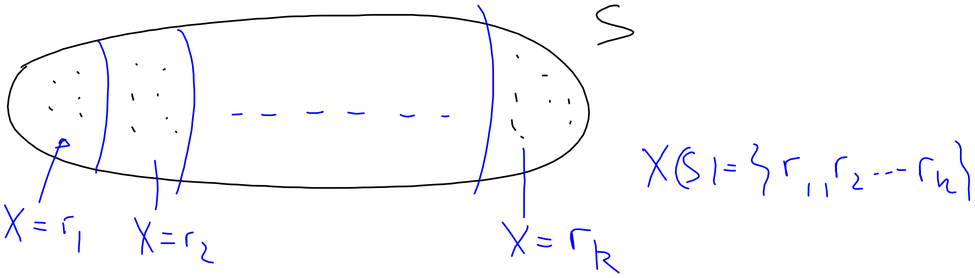
$X = \text{value} \quad E(X) = \frac{1}{6}(1+2+3+4+5+6)$

$= \frac{7}{2}$

$$E(X) = \sum_{s \in S} X(s) \cdot p(s)$$

$$= \sum_{r \in X(S)} r \cdot p(X=r)$$

where $X(S)$ is the set of values X can take on elements from S



Linearity of expectation

$X_1, X_2 \cdots X_n$ random variables on S

set $X = X_1 + X_2 + \cdots + X_n$

that is $X(s) = \sum_{i=1}^{n} X_i(s)$

Theorem $E(X) = E(X_1) + \cdots + E(X_n)$

P:

$$E(X) = \sum_{s \in S} X(s) \cdot p(s)$$

$$= \sum_{s \in S} (X_1(s) + \cdots + X_n(s)) p(s)$$

$$= \sum_{s \in S} X_1(s) \cdot p(s) + \sum_{s \in S} X_2(s) \cdot p(s) + \cdots + \sum_{s \in S} X_n(s) \cdot p(s)$$

$$= E(X_1) + \cdots + E(X_n)$$

Ex: 2 dice (roll) $X_i(s)$ = value of dice i $i = 1, 2$

$X = X_1 + X_2$ (sum of values)

Find $E(X)$

method 1:

$$E(X) = \sum_{r \in X(S)} r \cdot p(X=r)$$

$$= 2 \cdot p(X=2) + 3 \cdot p(X=3) + \dots + 12 \cdot p(X=12)$$

method 2 (use linearity of expectation)

$$E(X) = E(X_1) + E(X_2)$$

$$= \frac{7}{2} + \frac{7}{2} = 7$$

(since $E(X_i) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$

what if we roll 100 dice or 1000

still easy with metod 2 (contrary to method 1)

because if $n = 100$:

$$E(X) = 100 \cdot \frac{7}{2} = 350$$

Note that we also have

if Y is a random var on S

and a, b are constants, then

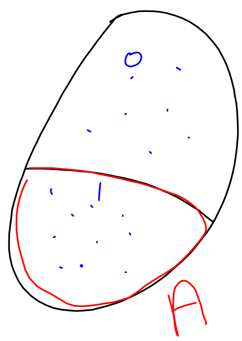
$$E(aY+b) = a\,E(Y)+b$$

Indicator random variables

Let A be an event in sample space S

and define the random var X_A as

$$X_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$



so $p(X_A = 1) = p(A) = p$

and (!) $E(X_A) = 1 \cdot p + 0 \cdot (1-p) = p$

Ex 6 Hatcheck problem

$\boxed{}_1 \quad \boxed{}_2 \quad - - - - \quad \boxed{}_n$ n hats

$\downarrow$ random permutation π

$\boxed{1} \quad \boxed{2} \quad \boxed{n}$ n persons

E_i : person i gets his hat $\left(p(E_i) = \frac{1}{n}\right)$

$$X_i = X_{E_i} = \begin{cases} 1 & \text{if } i \text{ gets his hat} \quad p = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Define X as

$$X = X_1 + X_2 + \cdots + X_n$$

then X is precisely # of people who get their own hat

so $E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{n} = n\frac{1}{n} = 1$

Thm 2 Let X = # successes in n independent Bernoulli exps with success prob p.
Then $E(X) = np$

recall p (exactly k success in n expts) is given by $b(k,n,p) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$

so $E(X) = \sum_{k=0}^{n} k \cdot b(k,n,p) = np$

we claim

Define $X_i = \begin{cases} 1 & \text{if exp } i \text{ is success} \\ 0 & \text{otherwise} \end{cases}$ $p(X_i = 1) = p$

Then $X = \sum_{i=1}^{n} X_i$ is # of successes

and $E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np$

Geometric distribution

X has a geometric distribution with prob p if $p(X=k) = (1-p)^{k-1} \cdot p$

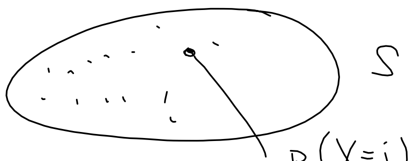
$$E(X) = \sum_{j=1}^{\infty} j \cdot P(X=j)$$

$$= \sum_{j=1}^{\infty} j \cdot (1-p)^{j-1} \cdot p$$

$$= p \sum_{j=1}^{\infty} j \cdot (1-p)^{j-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

note $$\sum_{j=1}^{\infty} (1-p)^{j-1} p = p \sum_{j=1}^{\infty} (1-p)^{j-1} = p \frac{1}{(1-(1-p))}$$

$$= p \cdot \frac{1}{p} = 1$$



$p(X=j) = (1-p)^{j-1} p$