Definition The random variables X and Y are called independt if

$$\forall r \in X(S)\ \forall t \in Y(S):\ p(X=r \wedge Y=t) = p(X=r)\,p(Y=t)$$

Thm 5 If X and Y are indep. then $E(XY) = E(X)\cdot E(Y)$

P:

$$E(XY) = \sum_{\substack{r\in X(S)\\ t\in Y(S)}} rt\,p(X=r\wedge Y=t)$$

by def of independence

$$= \sum_{\substack{r\in X(S)\\ t\in Y(S)}} rt\,p(X=r)\,p(Y=t)$$

$$= \Big(\sum_{r\in X(S)} r\,p(X=r)\Big)\Big(\sum_{t\in Y(S)} t\,p(Y=t)\Big)$$

$$= E(X)\,E(Y)$$

Ex 13 Flip a coin twice h/t

$$\left.\begin{array}{l} X = \#\text{ heads} \\ Y = \#\text{ tails} \end{array}\right\} \quad X+Y=2$$

$$p(X=2) = \frac{1}{4} = p(Y=0)$$

$$p(X=1) = \frac{1}{2} = p(Y=1)$$

$$p(X=0) = \frac{1}{4} = p(Y=2)$$

$$E(X) = 2\cdot p(X=2) + 1\cdot p(X=1) + 0\cdot p(X=0)$$

$$= 2\cdot\frac{1}{4} + 1\cdot\frac{1}{2} = 1$$

$$E(Y) = 1$$

$$E(XY) = 1\cdot p(X=1 \wedge Y=1) = 1\cdot\frac{1}{2} = \frac{1}{2}$$

so $E(XY) = \frac{1}{2} \neq 1 = E(X)E(Y)$

so X and Y are not independent

$\boxed{1} \quad \boxed{2} \quad \cdots \quad \boxed{n}$

$\downarrow$ random onto mapping

$\boxed{1} \quad \boxed{2} \quad \boxed{n}$

Def (Variance of a random variable)

X random variable on S

$$V(X) = \sum_{s\in S} (X(s) - E(X))^2 p(s)$$

standard deviation $\sigma(X) = \sqrt{V(X)}$

Thm 6 $V(X) = E(X^2) - (E(X))^2$

$$V(X) = \sum_{s\in S} (X(s) - E(X))^2 p(s)$$

$$= \sum_{s\in S} [X^2(s) - 2X(s)E(X) + (E(X))^2] p(s)$$

$$= \sum_{s\in S} X^2(s) p(s) - 2\sum X(s) E(X) p(s) + \sum (E(X))^2 p(s)$$

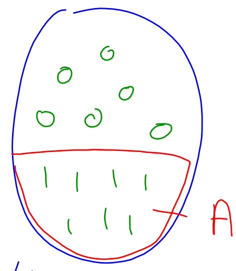
$$= E(X^2) - 2E(X) \sum_{s\in S} X(s) p(s) + (E(X))^2 \sum_{s\in S} p(s)$$

$$= E(X^2) - 2E(X)E(X) + (E(X))^2$$

$$= E(X^2) - (E(X))^2$$

Example variance of an indicator random variable.

Let $X_A(s) = \begin{cases} 1 & \text{if } s \in A \quad p \\ 0 & \text{if } s \notin A \quad (1-p) \end{cases}$



$E(X_A) = 1 \cdot p(X_A = 1) + 0 \cdot p(X_A = 0) = p(X_A = 1) = p$

$E(X_A^2) = 1 \cdot p(X_A^2 = 1) + 0 \cdot p(X_A^2 = 0)$

$= 1 \cdot p(X_A = 1) = p$

so $V(X_A) = E(X_A^2) - (E(X_A))^2 = p - p^2 = p(1-p)$

application n independent Bernoulli trials with succes probability p

$X =$ # successes in these n exp. $\quad b(k,n;p) = \binom{n}{k} p^k (1-p)^{n-k}$

$X_i = \begin{cases} 1 & \text{if } i\text{th exp. is success} \quad p \\ 0 & \quad (1-p) \end{cases}$

$X = \sum_{i=1}^{n} X_i$ so $E(X) = E\left(\sum_{i=0}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = n \cdot p$

$V(X) = E(X^2) - (E(X))^2$

Thm 7 if X and Y are independent, then

$V(X+Y) = V(X) + V(Y)$

Corollary with $X = \sum X_i$ $\quad X_i, X_j$ indep. when $(i \neq j)$

we have $V(X) = \sum_{i=1}^{n} V(X_i) = \sum_{i=1}^{n} p(1-p) = np(1-p)$

Chebyshev's inequality

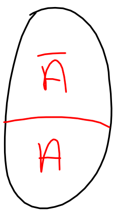
Let X be a random variable on S

then $p\left(|X(s) - E(X)| \geq r \right) \leq \dfrac{V(X)}{r^2}$

P: Define A as the event that

$$|X(s) - E(X)| \geq r$$

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s)$$



$$= \sum_{s \in A} (X(s) - E(X))^2 p(s) + \sum_{s \in S \setminus A} (X(s) - E(X))^2 p(s)$$

$$\geq \sum_{s \in A} r^2 \cdot p(s) + 0$$

$$= r^2 p(A)$$

so $V(X) \geq r^2 p(A)$

$\Downarrow$ $p(A) \leq \dfrac{V(X)}{r^2}$

application to coin flips.

suppose we flip a fair coin n times

X = heads so $E(X) = np = \frac{n}{2}$

$V(X) = np(1-p) = \frac{n}{4}$

so $P\left(\left|X - \frac{n}{2}\right| \geq \sqrt{n}\right)$ can be calculated:

$$\parallel$$

$$P\left(\left|X - E(X)\right| \geq \overset{r}{\sqrt{n}}\right) \leq \frac{V(X)}{r^2} = \frac{\frac{n}{4}}{(\sqrt{n})^2} = \frac{1}{4}$$

Probabilistic method (II)

First moment principle:

If $E(X) \le t$, then $p(X \le t) > 0$

P: suppose $E(X) \le t$ but $p(X \le t) = 0$

Then

$$t \ge E(X) = \sum_{r \in X(S)} r \cdot p(X=r)$$

$$= \sum_{\substack{r \in X(S) \\ r > t}} r \cdot p(X=r) + \sum_{\substack{r \in X(S) \\ r \le t}} r \cdot p(X=r)$$

$$= \sum_{\substack{r \in X(S) \\ r > t}} r \cdot p(X=r) + 0$$

$$> t \sum_{\substack{r > t \\ r \in X(S)}} p(X=r)$$

$$= t \quad \lightning$$

Markov's inequality

$$p(X \geq t) \leq \frac{E(X)}{t}$$

application k-SAT

$v_1, v_2, \ldots v_n$ 0,1 variables

$C = (v_{i_1} + \bar{v}_{i_2} + \ldots + v_{i_k})$ clause (of size k)

k literals (v_i or $\bar{v}_i$) for each clause

$(\bar{v}_i = 1 - v_i)$

C takes values 0 or 1 and it is 1 precisely when at least one of its literals are 1

e.g. $k=3$ $C = (v_1 + \bar{v}_3 + v_6)$

$v_1 = 0, v_3 = 0, v_6 = 1$

k-SAT problem $C_1, C_2 \ldots C_m$

I Given m clauses, each size k over the variables $v_1, v_2 \ldots v_n$

Q: can we find 0,1 assignment to variables s.t each C_i evaluates to 1 at the same time?

ex $k=3$ $(v_1 + \bar{v}_2 + v_3), (v_1 + v_2 + v_3)(v_2 + v_3 + \bar{v}_4)$

v_1, v_2, v_3, v_4 variables

Not easy in general! (NP-complete see DM553)

Theorem Every k-SAT instance with $m < 2^k$ clauses is satisfiable.

P: fix a random $\{0,1\}$ assignment to $v_1, \ldots v_n$

For each $i = 1, 2 \ldots m$ $X_i = \begin{cases} 1 & \text{if } C_i \text{ not ok} \\ 0 \end{cases}$

$X = \sum_{i=1}^{m} X_i$ # non satisfied clauses

$p(X_i = 1) = 2^{-k}$ so $E(X_i) = 2^{-k}$

$$E(X) = E\left(\sum_{i=1}^{m} X_i\right) = \sum_{i=1}^{m} E(X_i) = \sum_{i=1}^{m} 2^{-k} = \frac{m}{2^k} < 1$$

since $m < 2^k$

so $E(X) < 1$

so by Markovs ineq $p(X \geq 1) \leq \frac{E(X)}{1} < 1$