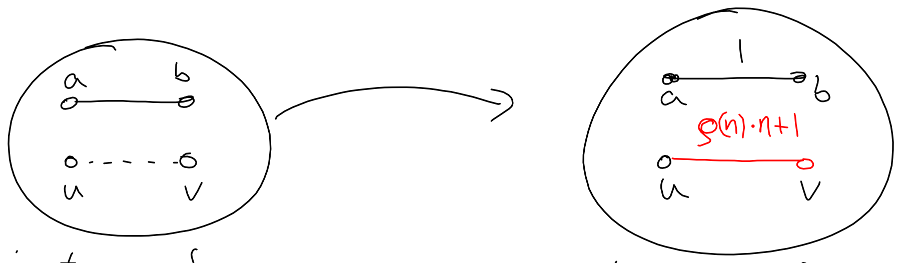
TSP without $\triangle$-inequality

suppose we have pol $\rho(n)$-approx alg $\mathcal{A}$

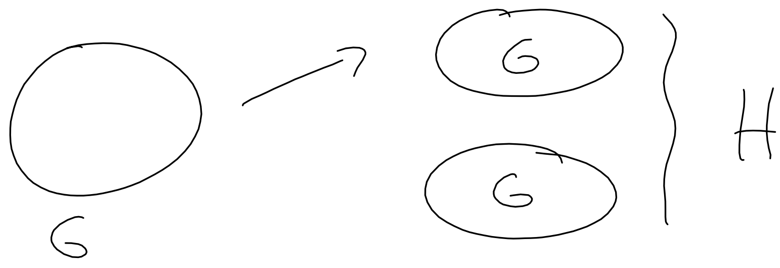


$G=(V,E)$ instance of HAMCYCLE $n=|V|$ (K_n, c)

<u>Observe</u> 1. G has Hamcycle $\Leftrightarrow$ opt TSP tour has cost n

2. Every TSP tour using an edge like uv has cost at least $\rho(n)\cdot n+1+(n-1)$
$= (\rho(n)+1)\,n > \rho(n)\cdot n$

$\Rightarrow$ We can solve HAMcycle by running $\mathcal{A}$ on (K_n, c) and returning 'yes' if cost of cycle from $\mathcal{A}$ is n and 'no' otherwise



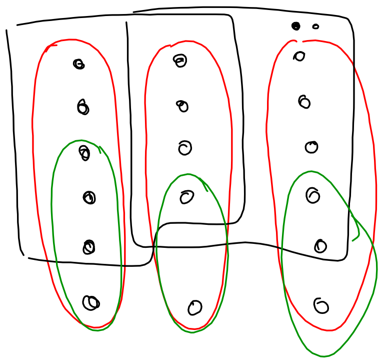
H has cycle covering
$\geq$ half its vertices
$\Updownarrow$
G is hamiltonian

Set cover problem. X is finite set

Input $(X, \mathcal{F})$ $\mathcal{F}$ family of subsets of S s.t $\bigcup_{S \in \mathcal{F}} S = X$

Output $\mathcal{F}' \subseteq \mathcal{F}$ s.t $\bigcup_{S \in \mathcal{F}'} S = X$ and $|\mathcal{F}'|$ is minimum.

Ex

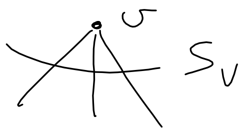


optimum = 3 (red sets)

Vertex cover is a special case of set cover.

$G = (V, E) \longrightarrow X = E$

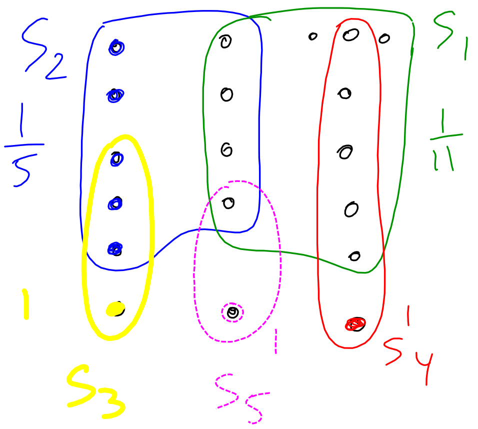
$\mathcal{F} = \{ S_v \mid v \in V \}$



VC NPC $\Rightarrow$ Set cover NPC as decision problem.

input $X, \mathcal{F}, K$

Quest $\exists$? sol' $\leq K$



Greedy set cover alg.
always pick set that
covers max no
uncovered vertices
until X is covered

sol from greedy alg has 5 set
optimum is 3.

How good is the greedy alg?

Thm greedy set cover is an $H(|X|)$-approx alg
where $H(n)=\sum_{i=1}^{n}\frac{1}{i}$

Proof: idea distribute the contribution (1) from each set in sol over the elements in the set as follows:

Let $S_1, S_2 \dots S_{|\mathcal{F}'|}$ be the output from alg

$\forall x \in X$ give x weight $c_x = \frac{1}{|S_i - (S_1 \cup \dots \cup S_{i-1})|}$

here $|S_i - (S_1 \cup \dots \cup S_{i-1})|$ is the number of elements covered for the first time by S_i

So (*) $|\mathcal{F}'| = \sum_{x \in X} c_x \leq \sum_{S \in \mathcal{F}^*} \sum_{x \in S} c_x$ $\mathcal{F}^*$ opt. sol

assume $\boxed{(\square)\ \sum_{x \in S} c_x \leq H(|S|) \quad \forall S}$

then (*) becomes $|\mathcal{F}'| \leq \sum_{S \in \mathcal{F}^*} \sum_{x \in S} c_x \leq \sum_{S \in \mathcal{F}^*} H(|S|)$

$\leq |\mathcal{F}^*| \, H(\max_{S \in \mathcal{F}} \{|S|\})$

So greedy is a $H(n)$ approx alg <u>if</u> ($\square$) is true.

<u>Show</u> (□): $\sum_{x\in S} c_x \leq H(|S|)$

Look at fixed $S\in\mathcal{F}$ and rewrite it weight as follows:

Let $u_i = |S-(S_1\cup\dots\cup S_i)|$ for $i=0,1,2,\dots k$ — when S is covered by $S_1 \dots S_k$ and $S-(S_1\cup\dots\cup S_{k-1})\neq\emptyset$

so u_i is the number of elements of S still uncovered after we have taken $S_1,S_2\dots S_i$. $(u_0=|S|)$

Clearly $u_{i-1}\geq u_i$ and $u_{i-1}-u_i$ elements of S are covered for the first time when S_i is added.

We also have $|S_i-(S_1\cup\dots\cup S_{i-1})|\geq \underbrace{|S-(S_1\cup\dots\cup S_{i-1})|}_{=u_{i-1}}$ as S_i was chosen in step i

<u>now</u>
$$\sum_{x\in S} c_x = \sum_{i=1}^{k}(u_{i-1}-u_i)\frac{1}{|S_i-(S_1\cup\dots\cup S_{i-1})|}$$
$$\leq \sum_{i=1}^{k}(u_{i-1}-u_i)\frac{1}{|S-(S_1\cup\dots\cup S_{i-1})|} = \sum_{i=1}^{k}(u_{i-1}-u_i)\frac{1}{u_{i-1}}$$

Note that for $b\geq a$ $H(b)-H(a)=\sum_{i=a+1}^{b}\frac{1}{i}\geq(b-a)\cdot\frac{1}{b}$

Thus $\sum_{x\in S} c_x \leq \sum_{i=1}^{k}(u_{i-1}-u_i)\cdot\frac{1}{u_{i-1}} \leq \sum_{i=1}^{k}(H(u_{i-1})-H(u_i))$

$$= H(u_0)-H(u_k)$$
$$= H(u_0) \quad \text{as } u_k=0$$
$$= H(|S|)$$

Indicator random variable for $\overset{\text{event}}{\vee}$ A

$$X_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & s \notin A \end{cases}$$

$$E(X_A) = p(A)$$

randomized
approx. alg for max 3-SAT

$\mathcal{A}$ is a randomized ϱ-approx alg if

$\max\left(\frac{c}{c^*}, \frac{c^*}{c}\right) \le \varrho$ when c is the expected value of sol from $\mathcal{A}$

<u>Max 3-SAT</u> Given 3-SAT instance $\mathcal{F}$ over variables $x_1 \ldots x_n$ on m clauses $C_1, \ldots C_m$

Find a truth assignment that satisfies maximum # of clauses

Not likely to be polynomial as this would imply 3-SAT $\in$ P so P=NP.

<u>Randomized alg $\mathcal{A}$</u>

For $i := 1$ to n

assign x_i value true or false randomly $\left(p(\text{true}) = p(\text{false}) = \frac{1}{2}\right)$

We show that if X is no of satisfied clauses by $\mathcal{A}$ then $E(X) = \frac{7}{8}m$

Clearly optimum is at most m

Hence we have $\frac{c^*}{c} \le \frac{m}{\frac{7m}{8}} = \frac{8}{7}$

need to show $E(X) = \frac{7}{8}m$

$X = X_1 + X_2 + \cdots + X_m$ where

$$X_i = \begin{cases} 1 & \text{if } C_i \text{ is satisfied} \quad p = \frac{7}{8} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \sum_{i=1}^{m} E(X_i) = \sum_{i=1}^{m} \frac{7}{8} = \frac{7m}{8}$$

Weighted vertex cover:

$G=(V,E)$ $\quad w: V \to \mathbb{R}_+$

Find VC of min weight.

NP version add $K \in \mathbb{R}_+$ and want VC of weight $\leq K$

NPC as $VC \leq_p$ this

Formulate as a 0-1 integer programming problem.

variable $X(v) = \begin{cases} 1 & v \text{ chosen} \\ 0 & \text{otherwise} \end{cases}$

conditions:

$$X(u)+X(v) \geq 1 \quad \forall uv \in E$$

objective $\quad \min \sum_{v \in V} w(v) \cdot X(v)$

$$\underset{\text{opt}}{\overset{Z_I}{\parallel}} = \min \sum w(v) X(v)$$

s.t $\quad X(u)+X(v) \geq 1 \quad \forall uv \in E$

$\quad X(u) \in \{0,1\} \quad \forall u \in V$

LP-relaxation

$\min \sum w(v) X(v) = Z_{LP}$

s.t $\quad X(u)+X(v) \geq 1 \quad \forall uv \in E$

$\quad 0 \leq X(u) \leq 1 \quad \forall u \in V$

$$Z_{LP} \leq Z_I = opt$$

LP relaxation can be solved in pol time.
as $\exists$ pol alg for lin. progr.

approximation alg B:

Solves LP problem and let

$\bar{X} = (\bar{x}_v)_{v \in V}$ be solution.

let $U = \{ v \in V \mid \bar{X}(v) \geq \frac{1}{2} \}$

Then U is a vertex cover:

$\bar{X}(u) + \bar{X}(v) \geq 1 \quad \forall\, uv \in E$

so either $u \in U$, $v \in U$ or $u, v \in U$

Why is the cost of U no more than $2 * opt$?

$$opt \geq Z_{LP} = \sum_{v \in V} \bar{X}(v) w(v)$$

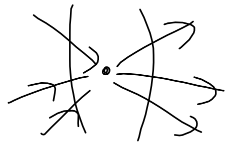
$$\geq \sum_{\{v \mid \bar{X}(v) \geq \frac{1}{2}\}} \bar{X}(v) w(v)$$

$$\geq \sum_{\{v \mid \bar{X}(v) \geq \frac{1}{2}\}} \frac{1}{2} w(v)$$

$$= \frac{1}{2} \sum_{v \in U} w(v) = \frac{1}{2} w(U)$$

$\Rightarrow w(U) \leq 2 * opt.$

D is eulerian if



$d^+(v)$

$\forall v : d^+(v) = d^-(v)$ $d^-(v)$

P1 Given D

Question does D have a spanning Eulerian subdigraph D' s.t $d^+_{D'}(v) > 0 \; \forall v \in V$?

P1 $\in$ P

P1' = P1 + we want D' to be connected

e.g.

not a sol to P1' but is a sol to P1

$\boxed{\text{P1' is NPC}}$