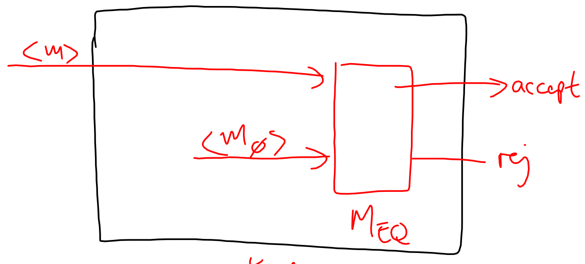
$EQ_{TM} = \{ \langle M_1\rangle\langle M_2\rangle \mid M_1, M_2$ are TM's s.t $L(M_1) = L(M_2) \}$

show that E_{TM} can be "reduced" to EQ_{TM}

Suppose M_{EQ} decides EQ_{TM}

$M_\emptyset$ is a TM s.t $L(M_\emptyset) = \emptyset$



M^* decides E_{TM}

<u>Def.</u> $L \leq_m L'$

'L is mapping reducible to L''

If $\exists f$ which is Turing-computable s.t

$x \in L$

$\Updownarrow$

$f(x) \in L'$

$x \xrightarrow{f} f(x)$

can be done by a TM

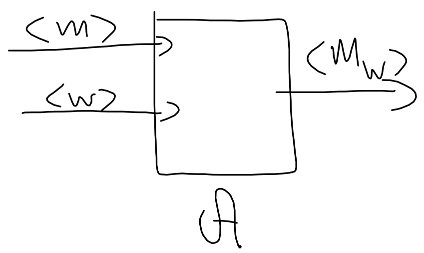
example $\langle M \rangle \xrightarrow{f} \langle M \rangle \langle M_\varphi \rangle$

is a mapping reduction from E_{TM} to EQ_{TM}

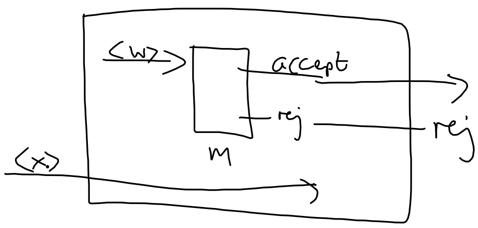
$H_\epsilon = \{\langle M\rangle \mid M$ is a TM and $\epsilon \in L(M)\}$

<u>Thm</u> H_ϵ is undecidable

P: we show that $A_{TM} \le_m H_\epsilon$



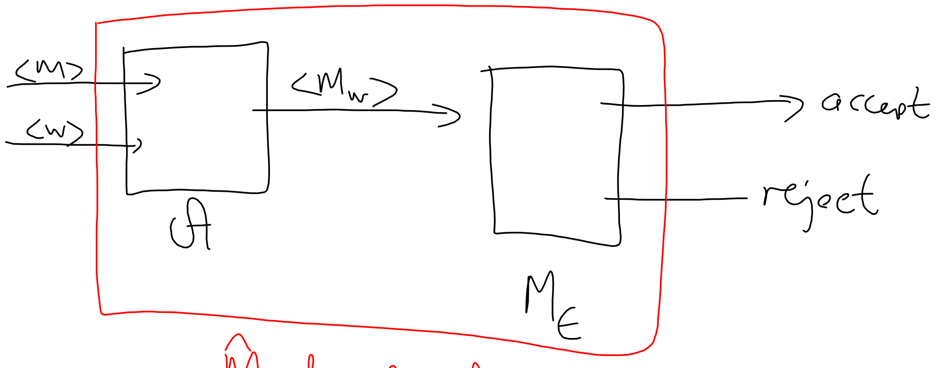
s.t. $L(M_w) = \begin{cases} \emptyset & \text{if } w \notin L(M) \\ \Sigma^* & \text{if } w \in L(M) \end{cases}$



so $A_{TM} \le_m H_\epsilon$ $\quad (x \in A_{TM} \Leftrightarrow f(x) \in H_\epsilon)$

this implies that if H_ϵ was decidable then A_{TM} would decidable. <u>Conclusion</u> H_ϵ is <u>not</u> decidable

By drawings:



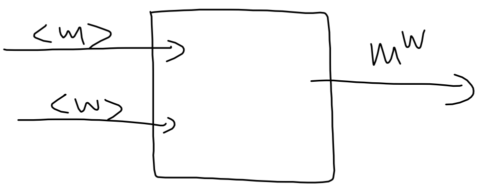
$\hat{M}$ decides A_{TM} }

$Regular_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$

Thm $Regular_{TM}$ is undecidable.

P: construct from an instance $\langle M\rangle\langle w\rangle$ of A_{TM} a TM M^w

s.t $L(M^w) = \begin{cases} \{a^n b^n \mid n \geq 0\} & \text{if } w \notin L(M) \\ \Sigma^* & \text{if } w \in L(M) \end{cases}$

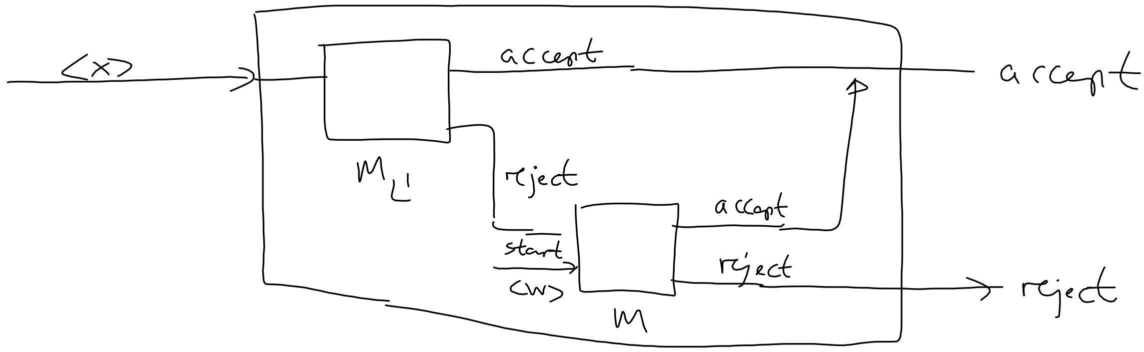


M^w: 1. Check whether input x is in $\{a^n b^n \mid n \geq 0\} = L'$

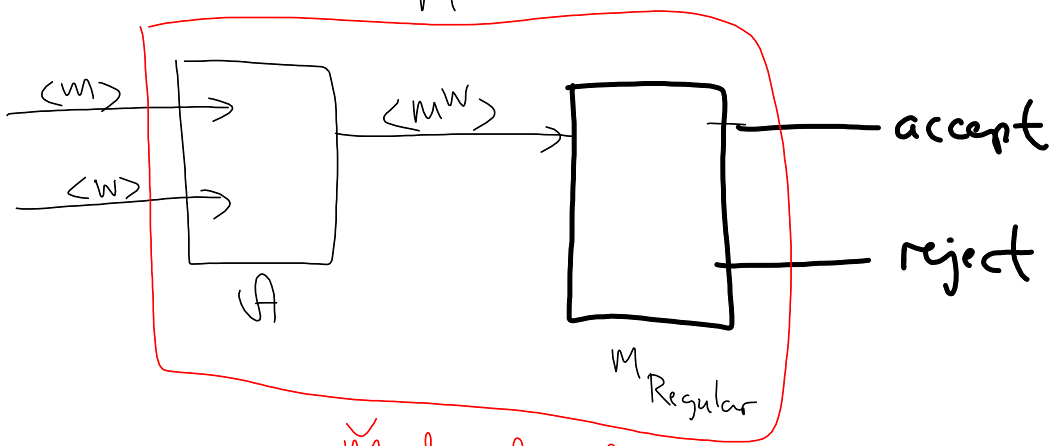
If 'yes' 'accept'

otherwise run M on w and accept x iff $w \in L(M)$

is



M^w



$\check{M}$ decides A_{TM}

$$\langle M\rangle\langle w\rangle \xrightarrow{f} \langle M^w\rangle$$

$$\langle M\rangle\langle w\rangle \in A_{TM} \iff \langle M^w\rangle \in \mathrm{Regular}_{TM}$$

A property concerns the language of a TM is it is about the set of strings accepted a TM:

ex Given $\langle M \rangle$

1. is $L(M)$ regular
2. is $L(M) = \emptyset$ $\quad E_{TM}$
3. $\exists ?\ x, y \in L(M)$ s.t $|x| = |y|$?
4. $\exists x \in L(M)$ s.t $|x| = 22$?
5. $\exists x \in L(M)$ s.t $|x| = i$ for $i = 1, 2 \ldots$

The following is not a question about the language of a TM:

$L_{all} = \{ \langle M \rangle \mid$ M is a TM and $\exists w$ s.t if started on w M will visit all its states except one $\}$

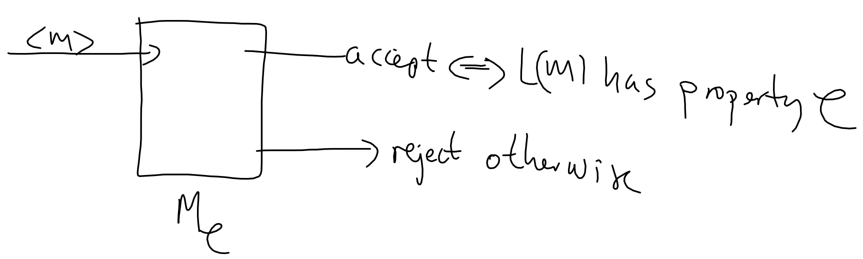
A property $\mathcal{C}$ of languages of TM's is non-trivial if there exist TM's M_1 and M_2 s.t. M_1 has property $\mathcal{C}$ and M_2 does not have property $\mathcal{C}$.

Thm (Rice's thm)

Every non-trivial property $\mathcal{C}$ about languages of TM's is undecidable.

P: w.l.o.g. we may assume that the empty language does not have property $\mathcal{C}$ (otherwise consider $\bar{\mathcal{C}}$)

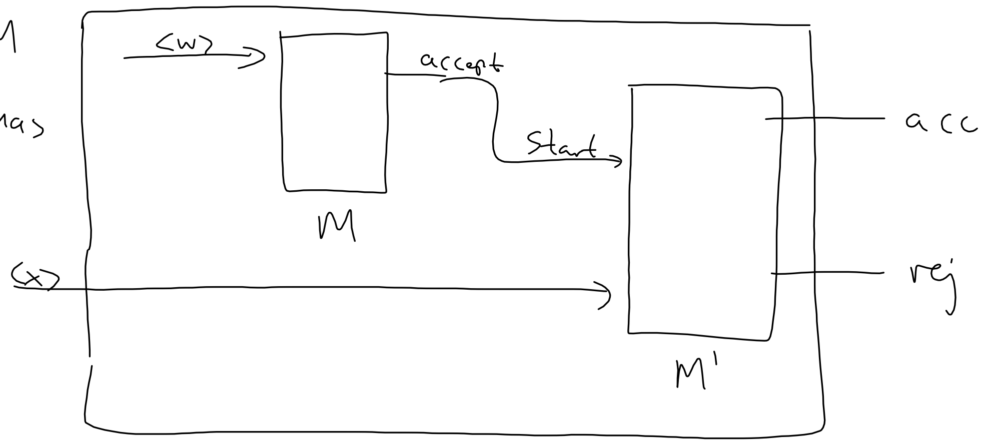
assume (for contradiction) that we have a TM $M_{\mathcal{C}}$:



Given $\langle M \rangle \langle w \rangle$ constuct $M(w)$ s.t

$L(M(w))$ has property $\mathcal{P}$ iff $w \in L(M)$.

M' is a TM s.t $L(M')$ has property $\mathcal{P}$



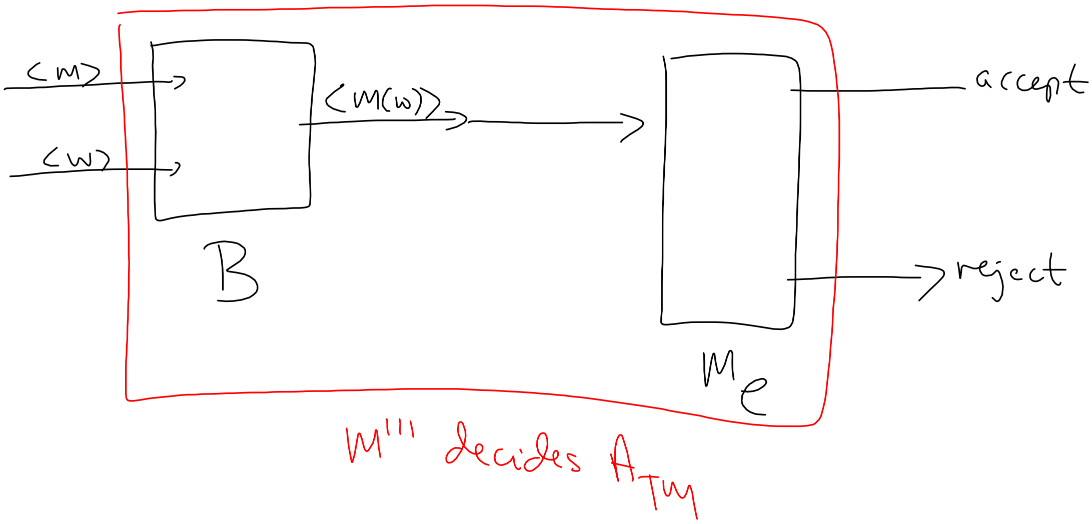
$M(w)$

$$L(M(w)) = \begin{cases} \emptyset & \text{if } w \notin L(M) \text{ or } M \text{ not a TM} \\ L(M') & \text{if } w \in L(M) \end{cases}$$

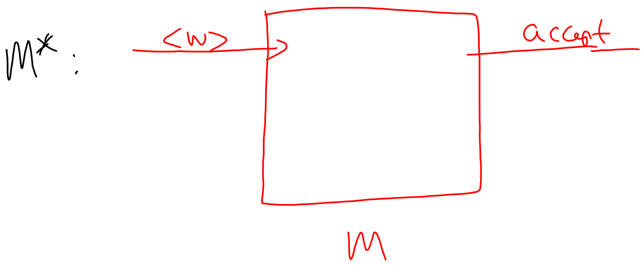
so $L(M(w))$ has property $\mathcal{P}$ iff $\langle M \rangle \langle w \rangle \in A_{TM}$

<u>Note</u> if M **is** not a TM then let

$M(w) = \mathbf{M}_{\emptyset}$


<M>
<w>
<M(w)>
B
accept
reject
Me
M''' decides ATM

new $L_{ALL} = \{ \langle M \rangle \mid M$ is a TM and if started on ε M will visit all its states except one $\}$



go to state q^* ← special state not used in simulation of M on w

↓ now visit all states via q^*

use special symbol $\tilde{a}$

$$\delta(q^*, -) = (q_1, \tilde{a}, R)$$

$$\delta(q_i, \tilde{a}) = (q_{i+1}, \tilde{a}, R)$$

until we reach q_{accept}/q_{rej}.

$$\langle M^* \rangle \in L_{ALL}$$

$$\Updownarrow$$

$$\langle M \rangle \langle w \rangle \in A_{TM}$$

So L_{ALL} is undecidable.

Properties of mapping reductions:

<u>Thm 5.22</u> $A \leq_m B$ and B is decidable

$\Downarrow$ A is decidable

<u>Cor 5.23</u> $A \leq_m B$ and A is undecidable

$\Downarrow$ B is undecidable

<u>also</u> If $A \leq_m B$ and B is recognizable the A is recognizable

Corollary: If $A \leq_m B$ and A is not recognizable then B is not recognizable.

$A_{TM} \leq_m \overline{E}_{TM} = \{ \langle w \rangle \mid$ either w is not a TM or w is a TM and $L(W) \neq \emptyset \}$

Given $\langle m \rangle \langle w \rangle$ construct $\langle M' \rangle$

s.t $L(M') = \begin{cases} \emptyset & \text{if } M \text{ is not TM or } M \text{ is a TM but } w \notin L(M) \\ \Sigma^* & \text{otherwise} \end{cases}$

$\overline{E}_{TM}$ is recognizable:

1. first check whether input is a TM M
2. let $w_1, w_2, w_3, \dots$ be a lex order of Σ^*
3. in round k simulate M for k steps on $w_1, w_2, \dots w_k$; if w_j is accepted by M we accept and stop

suppose $w_j \in L(M)$ and M accept w_j in p steps

then simulating for $\max\{j, p\}$ rounds will detect $w_j \in L(M)$

can use this to show that

E_{TM} is not recognizable

(otherwise E_{TM} would be decidable)