Cormen 35.5

NP: Subset sum $S = \{x_1, x_2, \ldots, x_n\}$, $t \in \mathbb{Z}_+$

Question: $\exists ?\ I \subseteq \{1, 2 \ldots n\}$ s.t

$$\sum_{i \in I} x_i = t \ ?$$

optimization version of subset sum

Find $t^* =$ maximum sum of value at most t

Definition A polyn. approximation scheme for a problem Q is an algorithm $A = A(\varepsilon)$ such A finds a solution which is within $(1+\varepsilon)$ of opt. sol to Q

$A(\varepsilon)$ runs in time polynomial in n (size of input) but not necessarily in ε.

e.g $n^{3/\varepsilon}$

A fully polynomial approx scheme:

as above but polynomial in n and $\frac{1}{\varepsilon}$

e.g $\left(\frac{1}{\varepsilon}\right)^7 \cdot n^3$

Solving subsetsum in exp time:

$S: x_1 \le x_2 \le \cdots \le x_n$ $S_i = \{x_1, \ldots, x_i\}$ $i = 0, \ldots, n$

$L_i =$ set of integers that can occur as
a sum of some of the elements in S_i

$L_0 = \{0\}$, $L_1 = \{0, x_1\}$, $L_2 \subseteq \{0, x_1, x_2, x_1 + x_2\}$

remove numbers $> t$

L_i add x_{i+1} to all numbers in L_i

L_{i+1}

$\vdots$

L_n

$|L_n|$ ~ can be up to 2^n

Ex $S = \{1, 5, 7, 8, 10 \ldots$

$L_0 = \{0\}$

$L_1 = \{0, 1\}$

$L_2 = \{0, 1, 5, 6\}$

$L_3 = \{0, 1, 5, 6, 7, 8, 12, 13\}$

see book for pseudo code

always remove sums that exceed t

running time?

can be up to 2^n as list could double

so <u>not</u> polynomial!

<u>But</u> polynomial if either

1. t is "small" as a function of n
2. or all x_i are at most $p(n)$ for some polynomial p.

Idea for approx scheme:

δ-trimming lists, i.e representing some values by a smaller one

if $\frac{y}{(1+\delta)} \leq z \leq y \iff y \leq z(1+\delta)$

trim

$L = \{0, 10, 11, 12, 15, 17, 18, 20, 24 \ldots\}$

$\delta = 0.1$

$L' = \{0, 10, 12, 15, 17, 20, 24 \ldots$

modified algorithm:

Form lists as before, except that we trim the new list before proceeding to (i+1)st step

Theorem The algorithm (Cormen 35.5) is a fully pol. approx scheme for subset sum (assuming we have chosen trim factor $\boxed{\delta = \frac{\varepsilon}{2n}}$)

P: we must show that $\frac{y^*}{z^*} \leq (1+\varepsilon)$,

where y^* is optimum solution and z^* is the maximum (rightmost/last) number in L'_n

If we removed y from the current list in step j then in L'_j y is represented by some z s.t

$$\frac{y}{(1+\delta)} \leq z \leq y$$

If z is then removed because of z'

then $\frac{z}{(1+\delta)} \leq z' \quad \wedge \quad \frac{y}{1+\delta} \leq z$

$$\Downarrow$$

$$\frac{\frac{y}{1+\delta}}{1+\delta} \leq z' \iff \frac{y}{(1+\delta)^2} \leq z'$$

In particular at the last iteration

y^* is represented by some element $z \in L'_n$

s.t $\frac{y^*}{(1+\delta)^n} \le z'$

but $z' \le z^* \Leftarrow$ max remaing element in L'_n

so $\frac{y^*}{(1+\delta)^n} \le z^* \iff \frac{y^*}{z^*} \le (1+\delta)^n = \left(1+\frac{\varepsilon}{2n}\right)^n$

$\downarrow$

$e^{\frac{\varepsilon}{2}}$

from calculus:

$$\left(1+\frac{\varepsilon/2}{n}\right) \to e^{\frac{\varepsilon}{2}} = 1+\frac{\varepsilon}{2}+\left(\frac{\varepsilon}{2}\right)^2+\cdots$$
$$\le 1+\varepsilon$$

so $\frac{y^*}{z^*} \le (1+\varepsilon)$

Running time? $L_n' = \{0, a_1, a_2, \ldots\ldots a_{r+1}\}$

The distance between a_i and a_{i+1} is at least a factor $(1+\delta)$: $a_{i+1} > (1+\frac{\varepsilon}{2n})a_i$ as a_{i+1} was not trimmed.

Thus $(1+\frac{\varepsilon}{2n})^r a_1 < a_{r+1} \leq t$

$\Rightarrow\ r < \log_{(1+\frac{\varepsilon}{2n})}(t)$

So L_n' has at most $r+2 < \log_{(1+\frac{\varepsilon}{2n})}(t)+2$

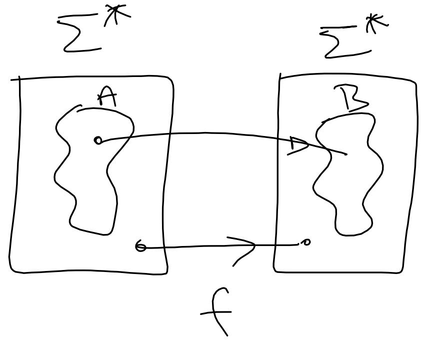
$= 2 + \frac{\ln t}{\ln(1+\frac{\varepsilon}{2n})}$ elements

$$\frac{\ln t}{\ln(1+\frac{\varepsilon}{2n})} \leq \frac{2n(1+\frac{\varepsilon}{2n})\ln t}{\varepsilon} \qquad \left(\frac{x}{1+x} \leq \ln(1+x) \Rightarrow \frac{1+x}{x} \geq \frac{1}{\ln(1+x)} \right.$$

So # elements in $L_n' \leq \frac{3n\ln t}{\varepsilon} + 2$ as $(1+\frac{\varepsilon}{2n}) \leq \frac{3}{2}$ $(0<\varepsilon<1)$

$= 3n \cdot \frac{1}{\varepsilon} \cdot \ln t + 2$ pol in $n, \ln t, \frac{1}{\varepsilon}$

reduction $A \leq_m B$ $\quad A' \leq_p B'$



$\exists$ TM that calculates $f(x)$

$\exists$ pol. alg to calculate x

$x \in A \iff f(x) \in B$

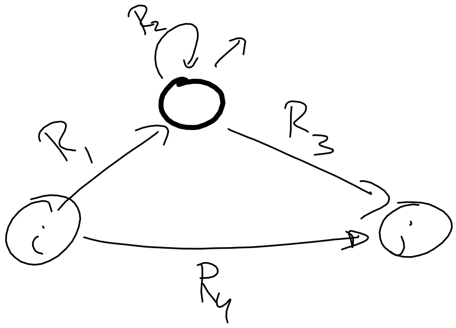
$x_1 \cdots x_n \; C_1 \cdots C_i \; C_m$

x

$\bar{x}$

s_1, s_2: $0 \; \cdots \; 0 \; 2 \; \cdots$

$1 \; 1 \; 1 \; 1 \; 1 \; 4 \; 4 \; \cdots \; 4$


R_2
R_1
R_3
R_4
i
j