Ahuja 2.51

$N = (V, A, \ell \equiv 0, u, b, c)$

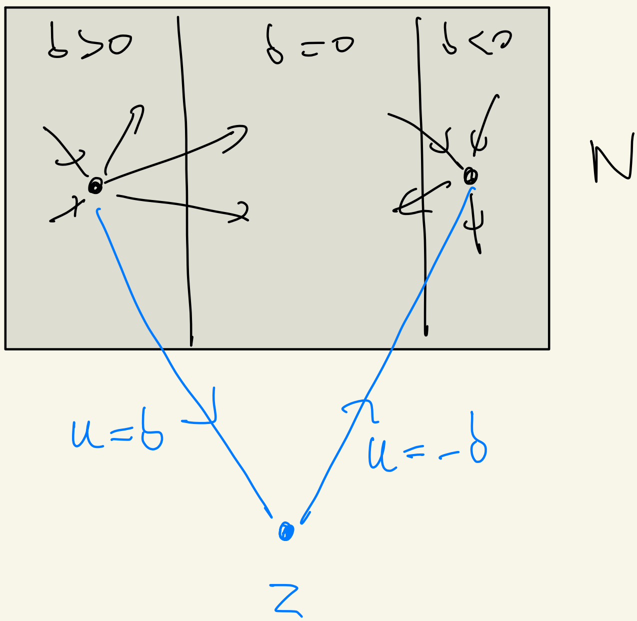
We want to find a flow x

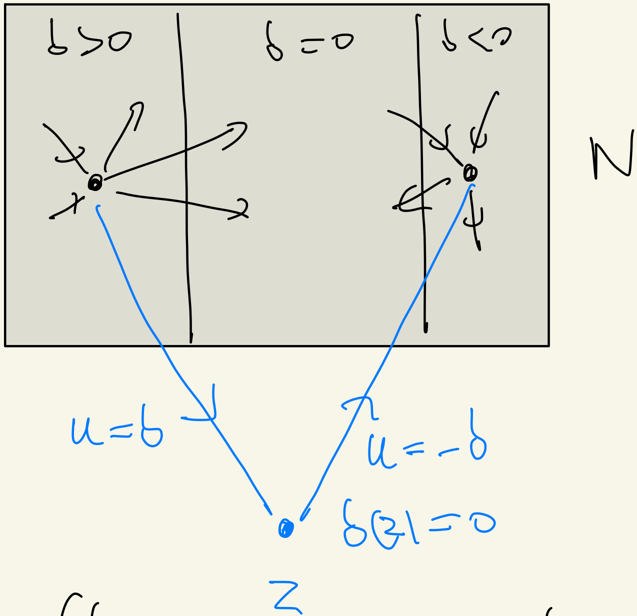
s.t

(1) $0 \leq b_x(v) \leq b(v)$ if $b(v) > 0$

(2) $0 \geq b_x(v) \geq b(v)$ if $b(v) < 0$

(3) $0 = b_x(v) = b(v)$ if $b(v) = 0$



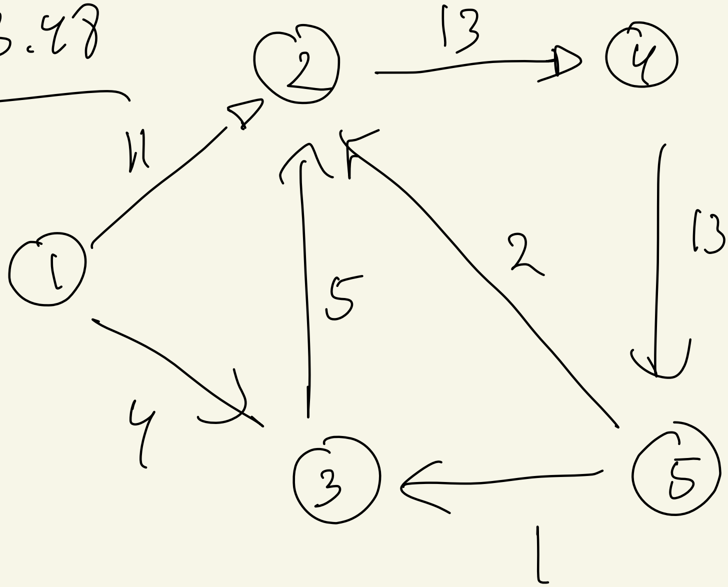


x flow in N satisfying (1)-(3)
send flow on blue arcs s.t
resulting flow x' has

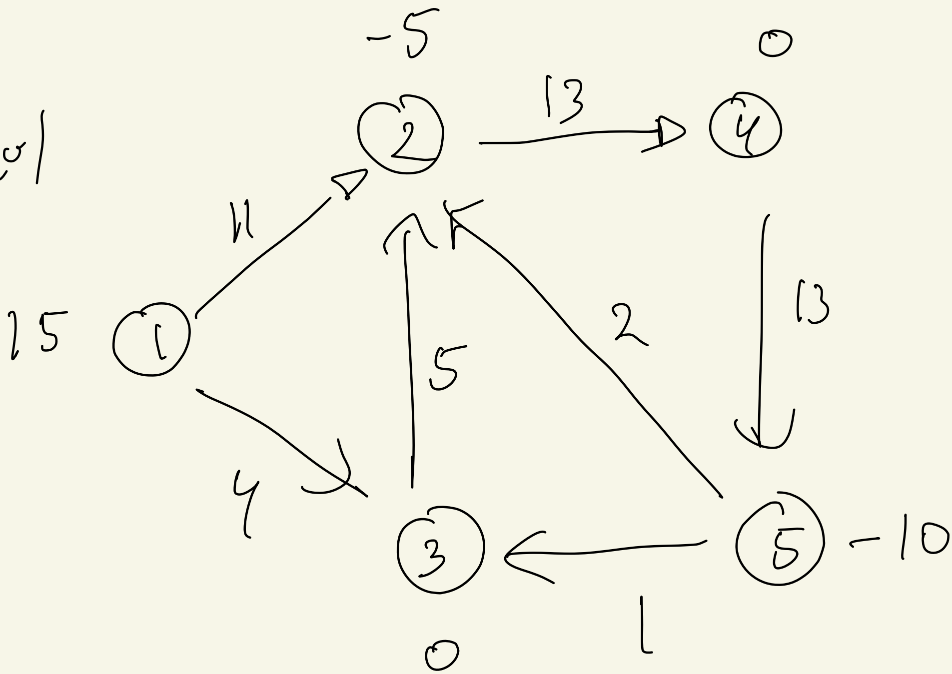
$$(*) \quad b_{x'}(v) = b_x(v) \quad \forall v \in V$$
$$\text{and} \quad b_{x'}(z) = 0$$

Conversely if x' satisfies (*)
we obtain good flow x by deleting
the vertex z

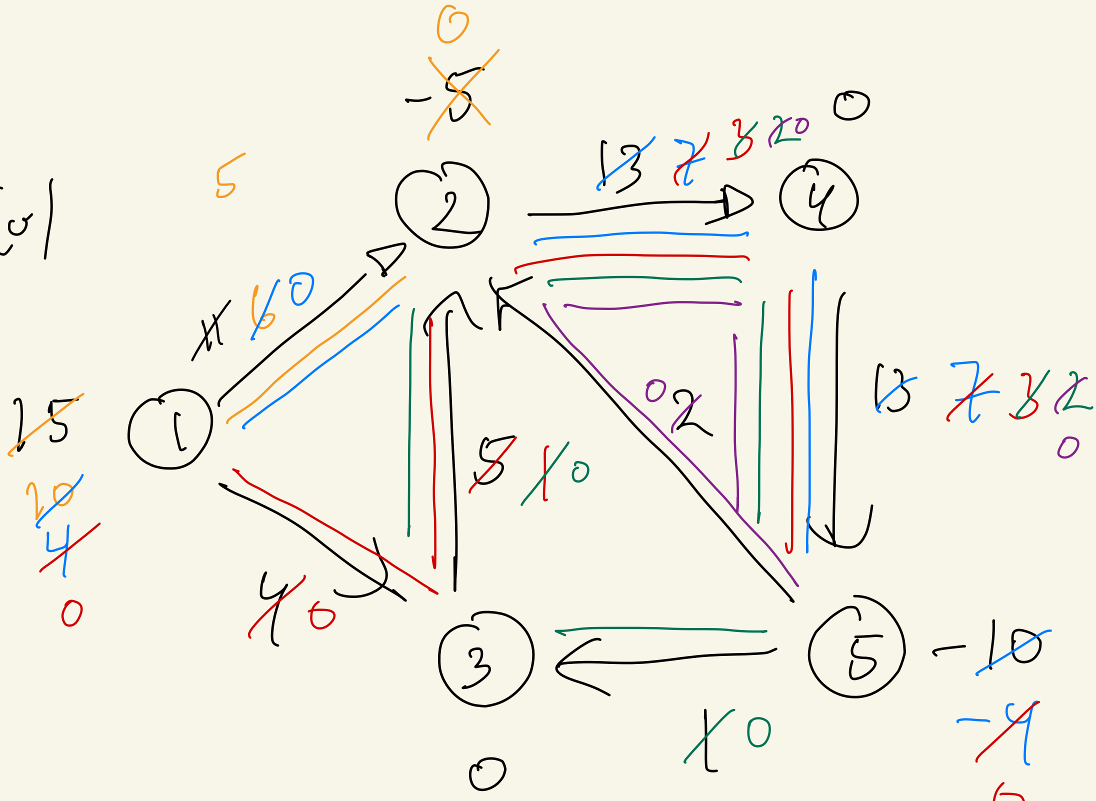
A 3.48



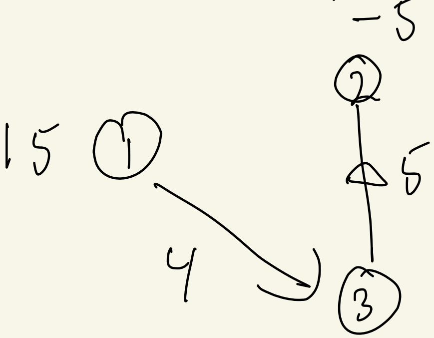
b(v)



b(v)



Not unique:



Ahuja 3.54

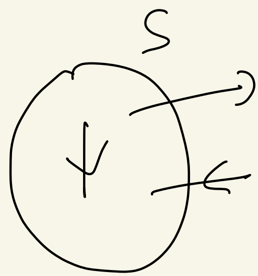
$N = (V, A, \ell, u)$

Claim x is a circulation in N

$\Updownarrow$

$$(*) \quad x(S, \bar{S}) - x(\bar{S}, S) = 0$$

$$\forall S \neq \emptyset, V \quad S \subseteq V$$



$\Downarrow$:

$$0 = \sum_{v \in S} b_x(v)$$

$$= x(S, \bar{S}) - x(\bar{S}, S)$$

$$\bar{S} = V - S$$

$\Uparrow$

$S = \{v\}$

$$b_x(v) = x(\{v\}, V - \{v\}) - x(V - \{v\}, \{v\})$$

$$= 0 \quad \text{by } (*)$$

Ahuja 3.53

$N = (V, A, \ell, u \equiv \infty)$

$D = (V, A)$ is connected

assume $\infty > \ell_{ij} > 0 \quad \forall ij$

show N has a feasible circulation

$\Updownarrow$

D is strongly connected

P:

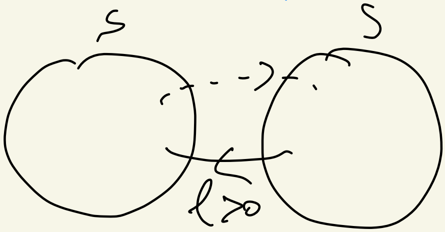
N has a feasible circulation

$\Updownarrow$ Hoffman's thm

$u(S, \bar{S}) \geq \ell(\bar{S}, S) \quad \forall S \subseteq V, \emptyset, V \neq S$

$\Updownarrow$

D is strongly connected



if no $S \to \bar{S}$ arc then $\ell(\bar{S}, S) > u(S, \bar{S}) = 0$

if D is strongly connected then $u(S, \bar{S}) \geq \ell(\bar{S}, S)$ $\forall S \neq \emptyset, V$

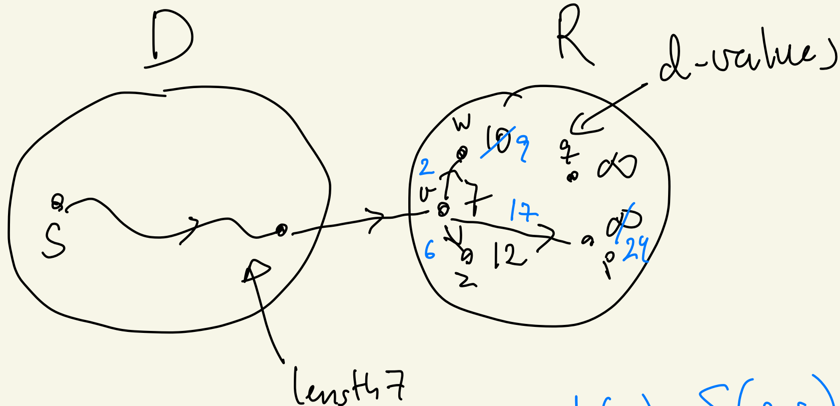
Ahuja 4.37 Max capacity paths

recall Dijkstras algorithm

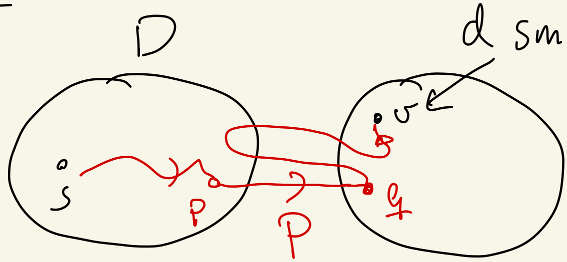
seek: $\delta(s,v)$ = length of shortest (s,v)-path $\quad \boxed{c_{ij} \geq 0 \quad ij \in A}$

initialization $\quad d(v) \leftarrow 0$ if $v = s$

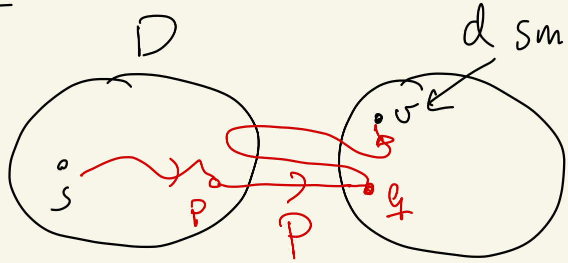
$d(v) \leftarrow \infty$ if $v \neq s$



why correct? invariant $\forall v \in D \quad d(v) = \delta(s,v)$



why Correct? invariant $\forall v \in D \quad d(v) = \delta(s,v)$



d smallest but $d(v) > \delta(s,v)$

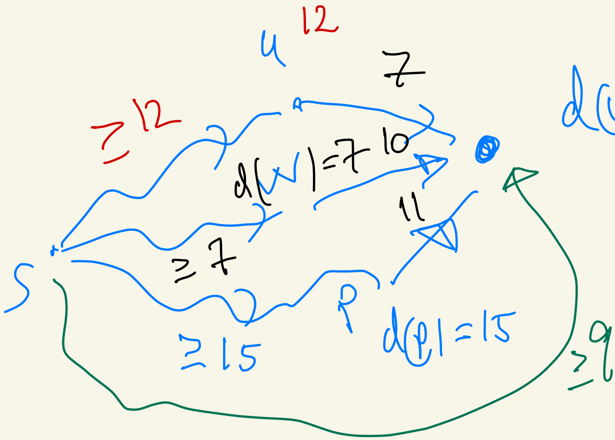
length of P is $\delta(s,v)$

assumed

$$\delta(s,v) < d(v) \le d(q)$$
$$= \delta(s,q)$$
$$\le \delta(s,v)$$

In our case we need to maximize capacity of paths

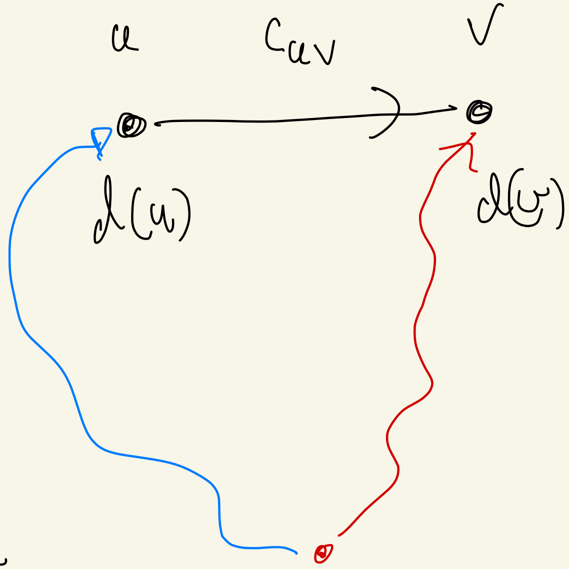
$d(v)$ is max cap of (s,v)-path found so far



$d(v) = \cancel{9}\ 11$

$\underline{\text{init}}$ $d(s) = \infty$

$d(v) = -\infty$

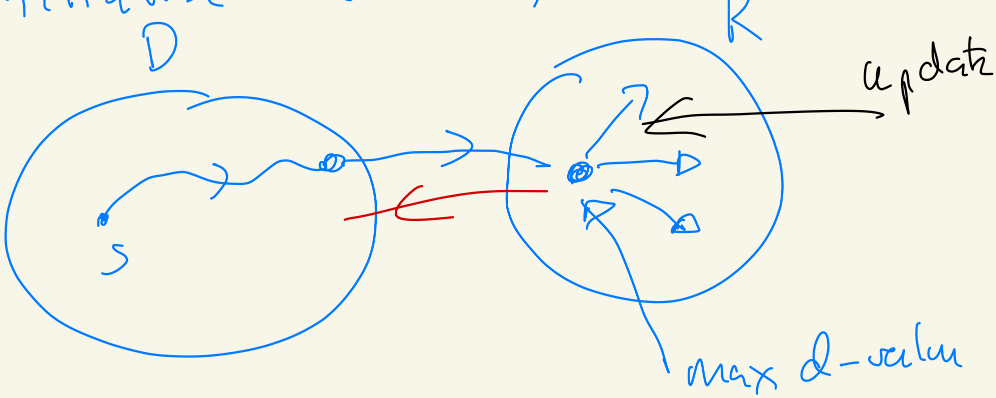


$\underline{\text{update}}$

$$d(v) \leftarrow \max\{d(v), \min\{d(u), c_{uv}\}\}$$

Modified Dijkstra:

initialize $d(s) \leftarrow \infty$, $d(v) \leftarrow -\infty$ $v \neq s$

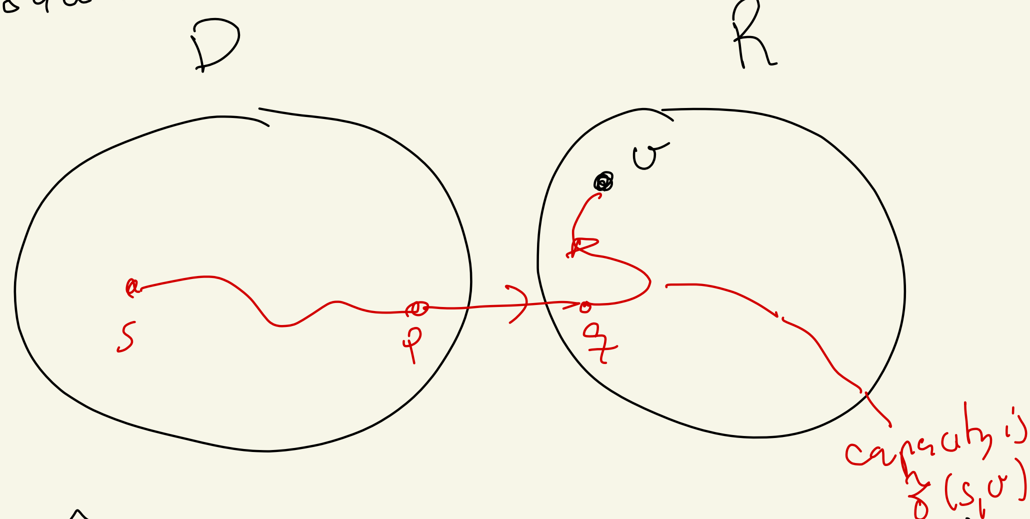


Invariant

$\forall v \in D \quad d(v) = \hat{\delta}(s, v)$

when $\hat{\delta}(s, v)$ = max cap of (s, v)-path

Suppose inv is false and v is first bad vertex moved to D

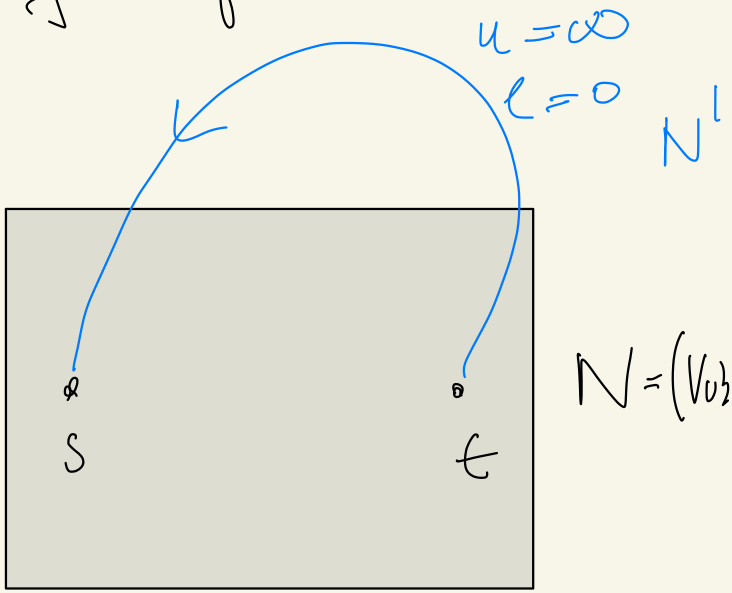


$$\hat{\delta}(s,v) > d(v) \geq d(q) \geq \hat{\delta}(s,v)$$

so invariant holds.

BJG 3.11

Eliminating lower bounds in max flow problems.



$N = (V \cup \{s,t\}, A, l, u)$

N' has a feasible circulation

$\Leftrightarrow$

N has a feasible (s,t)-flow

Given the network N' we eliminate lower bounds (as usual)

$$b(i) \; \xrightarrow[l_{ij}]{u_{ij}} \; b(j)$$

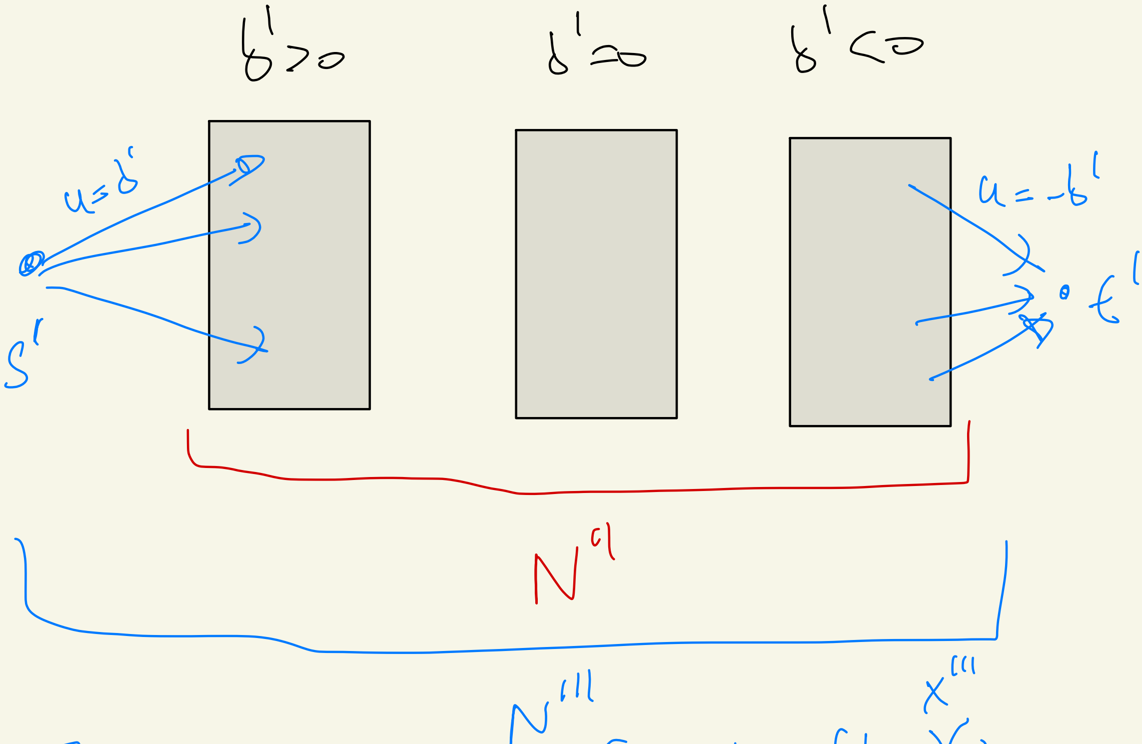
$$\Big\downarrow$$

$$b(i)-l_{ij} \; \xrightarrow[u_{ij}-l_{ij}]{} \; b(j)+l_{ij}$$

after removing <u>all</u> lower bounds, resulting balances are

$$b'(v) = \sum_{pv\in A} l_{pv} - \sum_{vq\in A} l_{vq}$$

Call new network N''

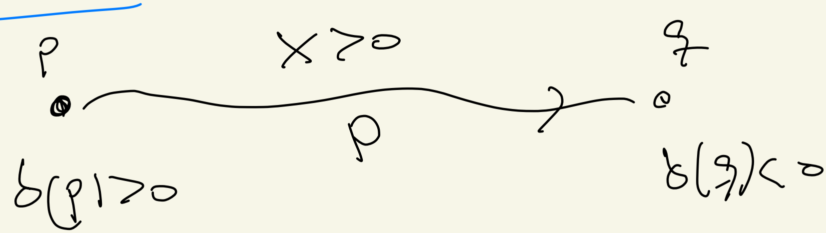


Find a maximum (s', t')-flow in

$$N''' = (V \cup \{s,t\} \cup \{s',t'\}, A', \ell \equiv 0, u'')$$

if $b_{x'''}(s') < \sum_{b'>0} b'(v)$ then no solution in N'' and hence none in N' and in N

BJG 3.15

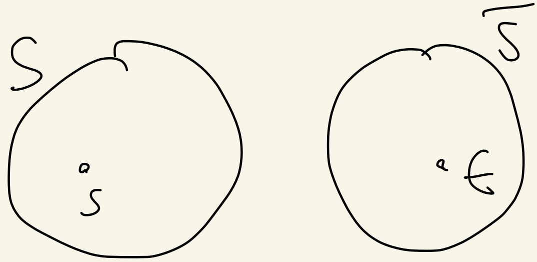


Suppose we ignore $b(p)$ and $b(q)$ and just take a path flow with flow $\delta(P)$ = min cap of arc on P

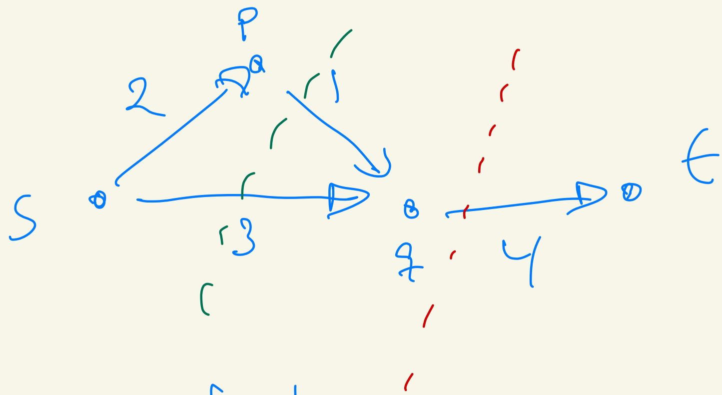
at least one new arc will get flow value zero in each iteration (removing a path or a cycle flow)

In total at most m paths + cycles

BJG 3.16   min (s,t)-cuts



(a)



(a) is false.

$$N = (V \cup \{s,t\}, A, \ell \equiv 0, u) \qquad k \text{ fixed}$$

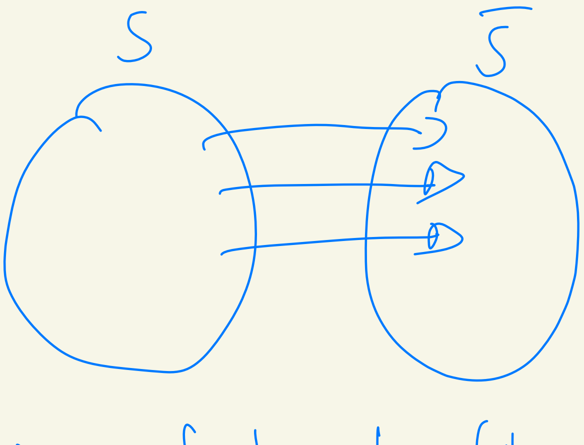
$$\downarrow$$

$$\bar{N} = (V \cup \{s,t\}, A, \ell \equiv 0, u') \qquad u'_{ij} = k u_{ij}$$

(b)

$u'(S, \bar{S}) = k \cdot u(S, \bar{S})$ so min cuts are preserved

(c) $u'_{ij} \leftarrow u_{ij} + k$ k fixed



(c) is false by the example in

(a)

BJG 3.18

Show that the Ford-Fulkerson algorithm will always terminate when capacities are rational numbers.

We know that FF-alg. terminates when all cap. are <u>integers</u>

Write all capacities as $u_{ij} = \frac{p_{ij}}{q_{ij}}$

$p_{ij}, q_{ij} \in \mathbb{Z}$

and Let $K = \prod_{ij \in A} q_{ij}$ and

$u'_{ij} = K \cdot u_{ij}$ now FF-alg takes a finite # of steps in N' and hence in N. $(\square)$

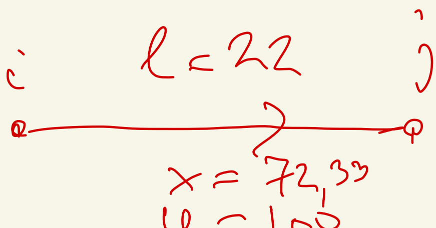
BJG 3.27 rounding an (s,t)-flow

$N=(V,A,l,u)$, x feasible in N,

assume that all l_{ij} and u_{ij} are integers

(a) prove that $\exists$ feasible integer (s,t)-flow x' in N such that

$$|x'_{ij}-x_{ij}|<1 \quad \forall ij\in A$$



replace l_{ij} by $l'_{ij}=\lfloor x_{ij}\rfloor$ and

u_{ij} by $u'_{ij}=\lceil x_{ij}\rceil$

then x is feasible in $N'=(V,A,l',u')$

By the integrality theorem there exist a feasible integer ~~flow~~ circulation x' in N' (*)

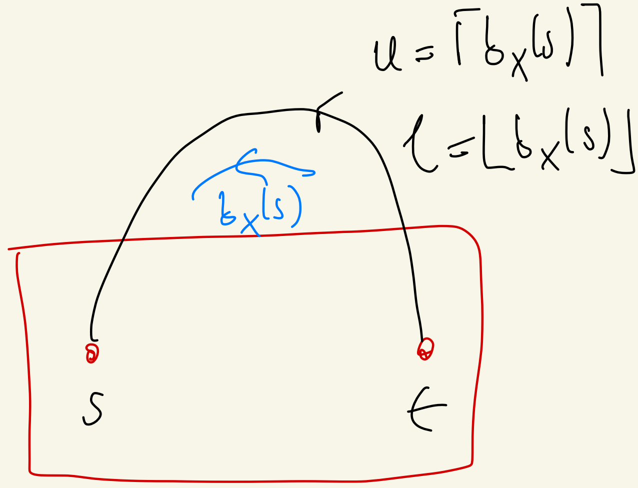
note that

$$|x'_{ij} - x_{ij}| < 1$$

since $x'_{ij} = l'_{ij}$ or $x'_{ij} = u'_{ij}$

Correction

N' is



(*) original x plus sending $b_x(s)$ along the arc ts is a feasible circulation in N'

(b) suppose we know that

$$|x| = b_x(t) \in \mathbb{Z}_+ \cup \{0\}$$

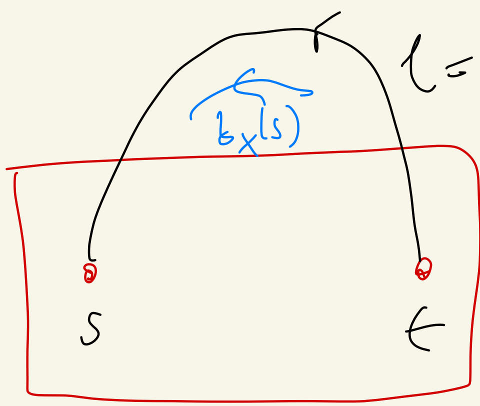
Show that there exists feasible integer flow x'' such

$$|x''| = |x|$$

$u = \lceil b_x(t) \rceil$

$l = \lfloor b_x(s) \rfloor$

equal



so the flow x' we found in (a) corresponds to integer valued flow x'' in N with $|x''| = |x|$

BJG 3.31

Show that using one maxflow calculation we can either find a feasible circulation in $N = (V, A, \ell, u)$ or decide that no feasible circulation exists

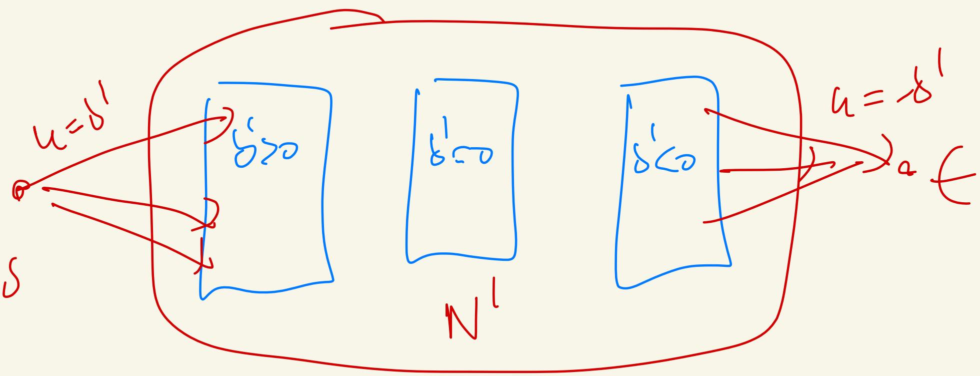
1) eliminate lower bounds

new balances

$$b'(i) = \sum_{ji \in A} \ell_{ji} - \sum_{ij \in A} \ell_{ij}$$

2) transform into a maxflow problem

3) find maxflow and use this to decide the existence of a solution and produce one when it exists.



Special case

if $b'(v)=0$ $\forall v$

then just take $x'\equiv 0$ in N'

in N this corresponds to taking $x_{ij}=l_{ij}$ $\forall ij\in A$

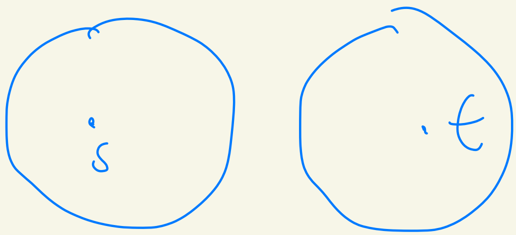
BJG 3.24 = Ahuja 6.35 with $u_{ij} \in \mathbb{Z}$ $\forall ij$

x is a maximum flow in N

(a) $u_{ij} \to u_{ij}+k$ for one arc N'

(b) $u_{ij} \to u_{ij}-k$ — — " —

can only get a larger flow in (a) and only if ij crosses all min cuts.



(a) update $N(x)$ according to change in u_{ij} ($r_{ij} \to r_{ij}+k$)

Find a max flow y in new $N(x)$

set $x' = x \oplus y$ then x' is a max (s,t)-flow in N'

can find y in time $O(km)$

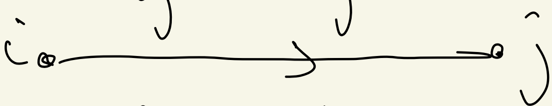
(b) $u'_{ij} \leftarrow u_{ij} - k$

Case 1 (easy) $x_{ij} \le u_{ij} - k$

x is still feasible and hence is maximum in N' (new network)

Case 2

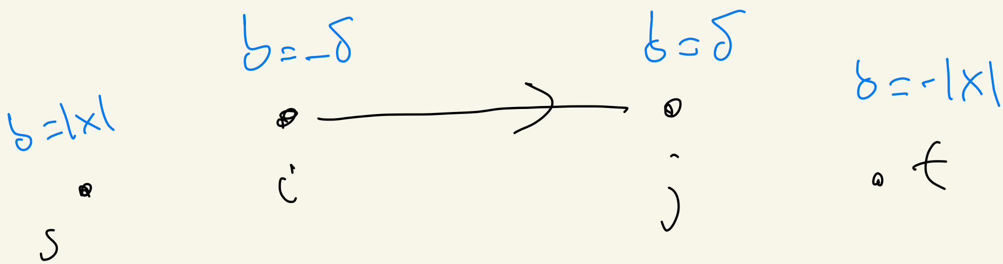
$$x_{ij} = u'_{ij} + \delta \qquad \delta > 0$$



make new flow x':

$$x'_{ij} \leftarrow u'_{ij} \quad \text{and} \quad x'_{pq} = x_{pq}$$

for all other arcs



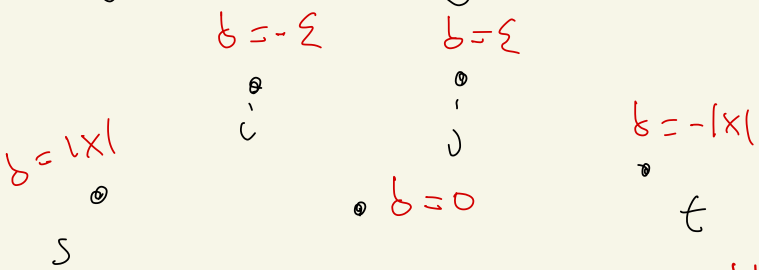
look at $N'(x')$.

If y is an (i,j)-flow in $N(x)$
then $z = x' \oplus y$ is a feasible flow in N'
with $b_z(v) = b_{x'}(v) + b_y(v)$

if $\exists$ (i,j)-flow y of value δ in $N(x')$
then 'add' this flow to x' and
we get a feasible (s,t)-flow z
of value $|z| = |x|$ so z is a
new max flow.
Can find y or determine that no
such flow exists in time $O(km)$

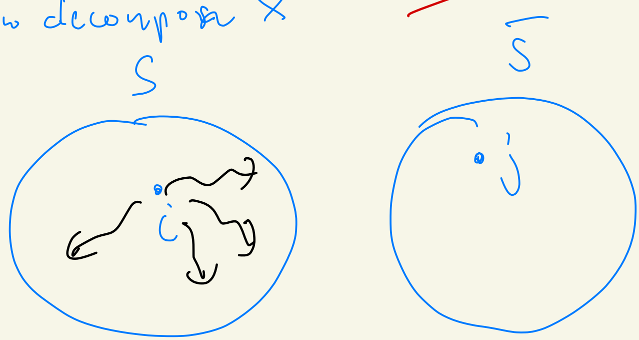
Suppose that the maximum flow value from i to j in $N(x')$ is $\delta - \varepsilon$ for some $\delta \geq \varepsilon \geq 0$

Let $x'' = x' \oplus y$ for such a flow y



Flow decompose x''

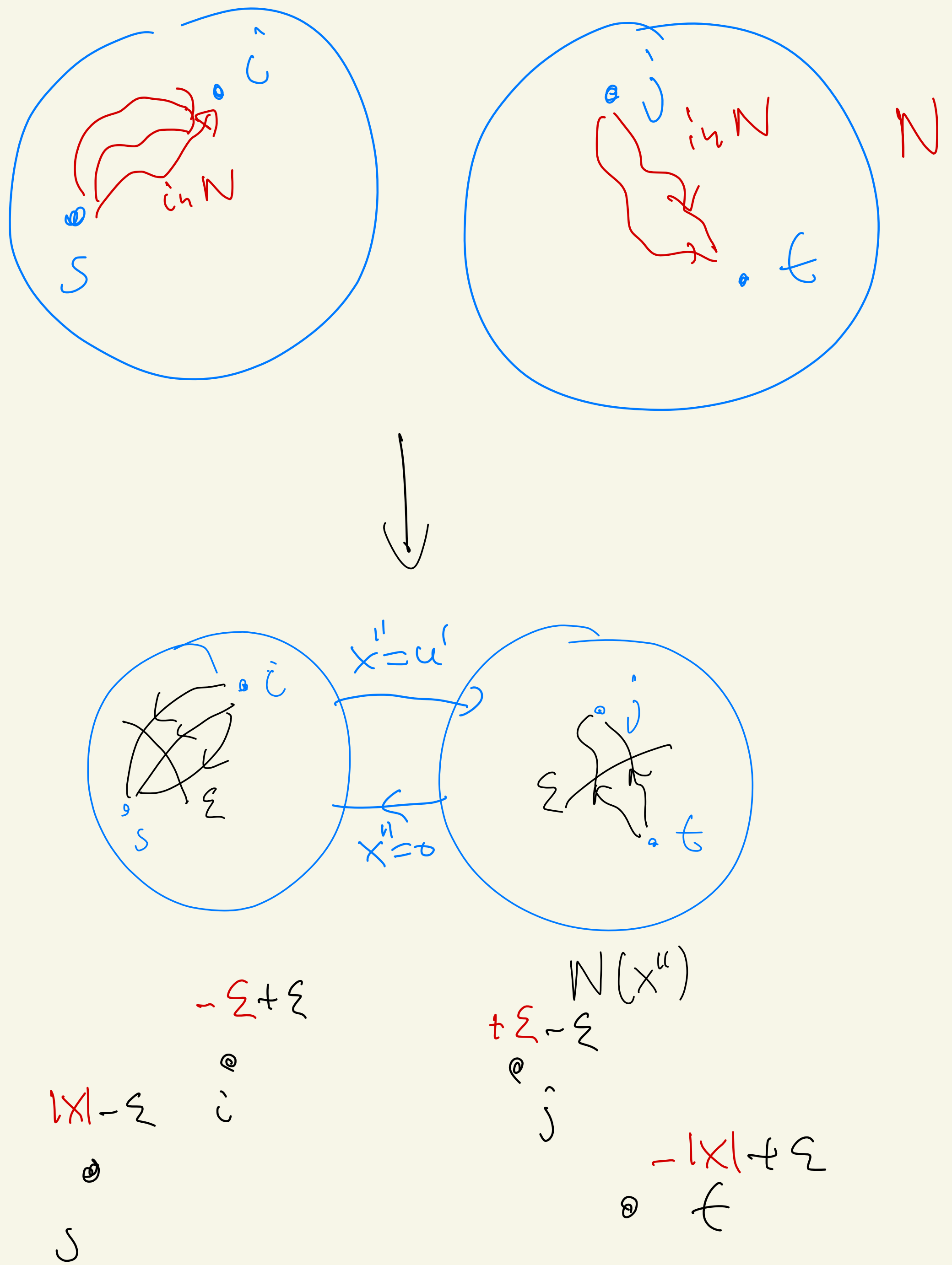
red paths



min cut in $N(x')$

(i, j)

$$S = \{ v \mid \exists\, (i,v)\text{-path in } N(x') \}$$



i
s
in N
j
in N
t
N
x'' = u'
x'' = 0
ε
ε
N(x'')
−ε+ε
+ε−ε
i
j
|X|−ε
s
−|X|+ε
t