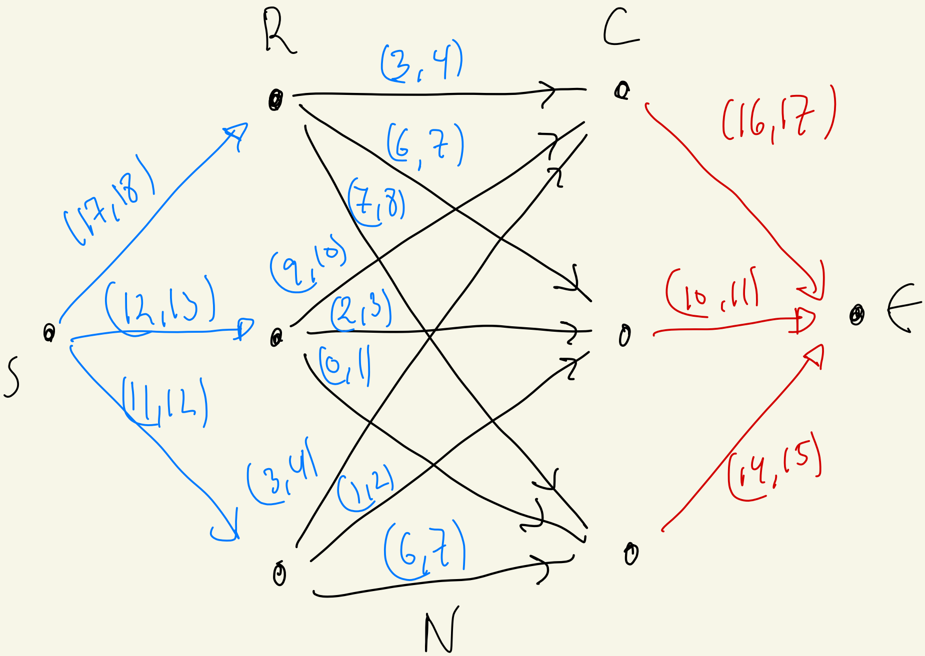
Ahuja appl 6.3 matrix rounding

C

x

R				
	3.1	6.8	7.3	17.2
	9.6	2.4	0.7	12.7
	3.6	1.2	6.5	11.3
	16.3	10.4	14.5	

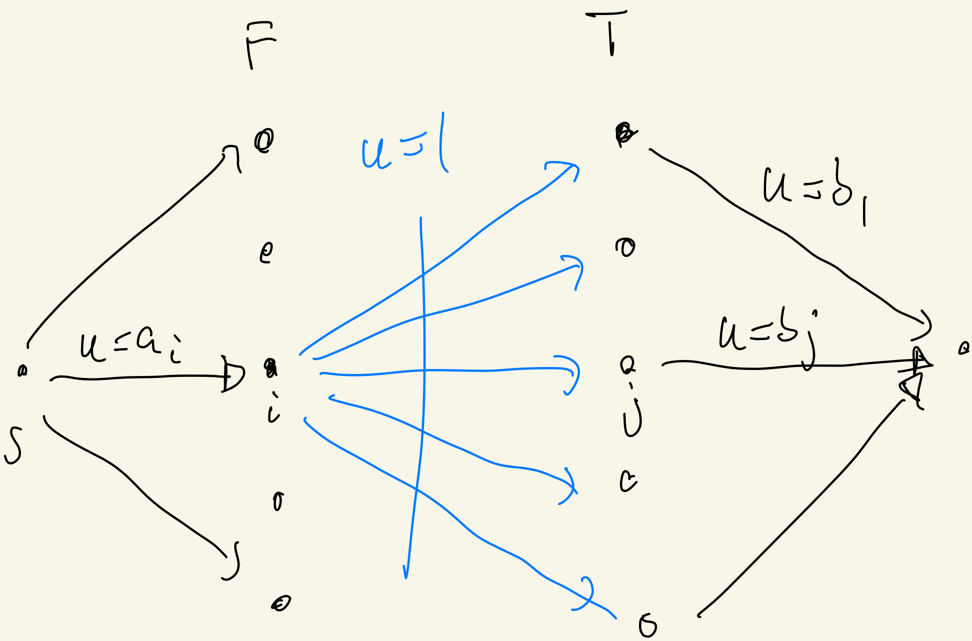


N has a feasible flow $\Leftrightarrow$ $\exists$ good rounding

Integrality thm + $\bar{x}$ $\Rightarrow$ feasible integer flow

Ahuja 6.1

seating families at tables
no two of same family at same
table.



$\exists$ x flow of value $\sum_{i \in F} a_i$ (x integer)

$\Updownarrow$

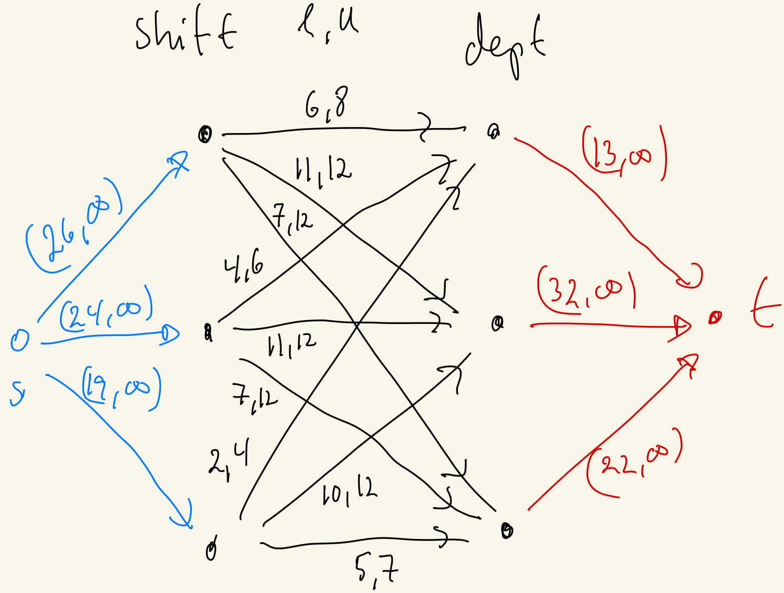
$\exists$ good seating $\qquad (\sum a_i \le \sum b_j)$

Ahuja 6.2

departments

shifts				
	[6,8]	[11,12]	[7,12]	≥ 26
	[4,6]	[11,12]	[7,12]	≥ 24
	[2,4]	[10,12]	[5,7]	≥ 19
	≥ 13	≥ 32	≥ 22	

N



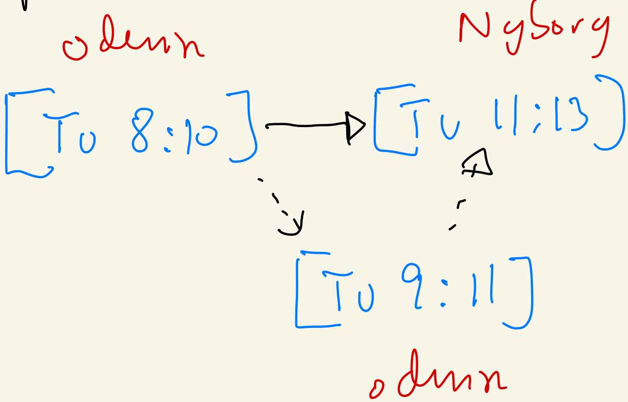
claim $\exists$ feasible integer flow in N

$\Updownarrow$

$\exists$ good schedule

Ahuja 6.41

optimal coverage of sports events



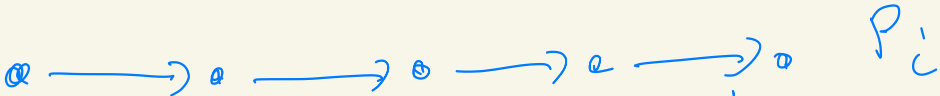
obtain an acyclic digraph D

claim D has a path cover with k paths

$\Updownarrow$

we can cover all events with k reporters k reporters enough

$\downarrow$

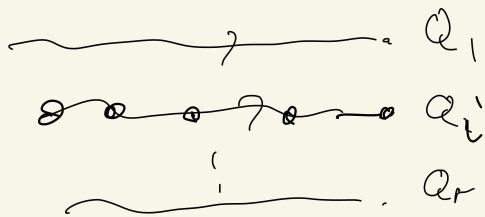


events covered by reporter i

P_i is a path in D

$P_1 \dots P_k$ cover $V(D)$ ✓

suppose $Q_1 \dots Q_r$ cover $V(D)$



assign events corresponding to $V(Q_i)$
to reporter i

Conclusion

We can solve the problem by finding a minimum path cover in D

This is a minimum value flow problem.

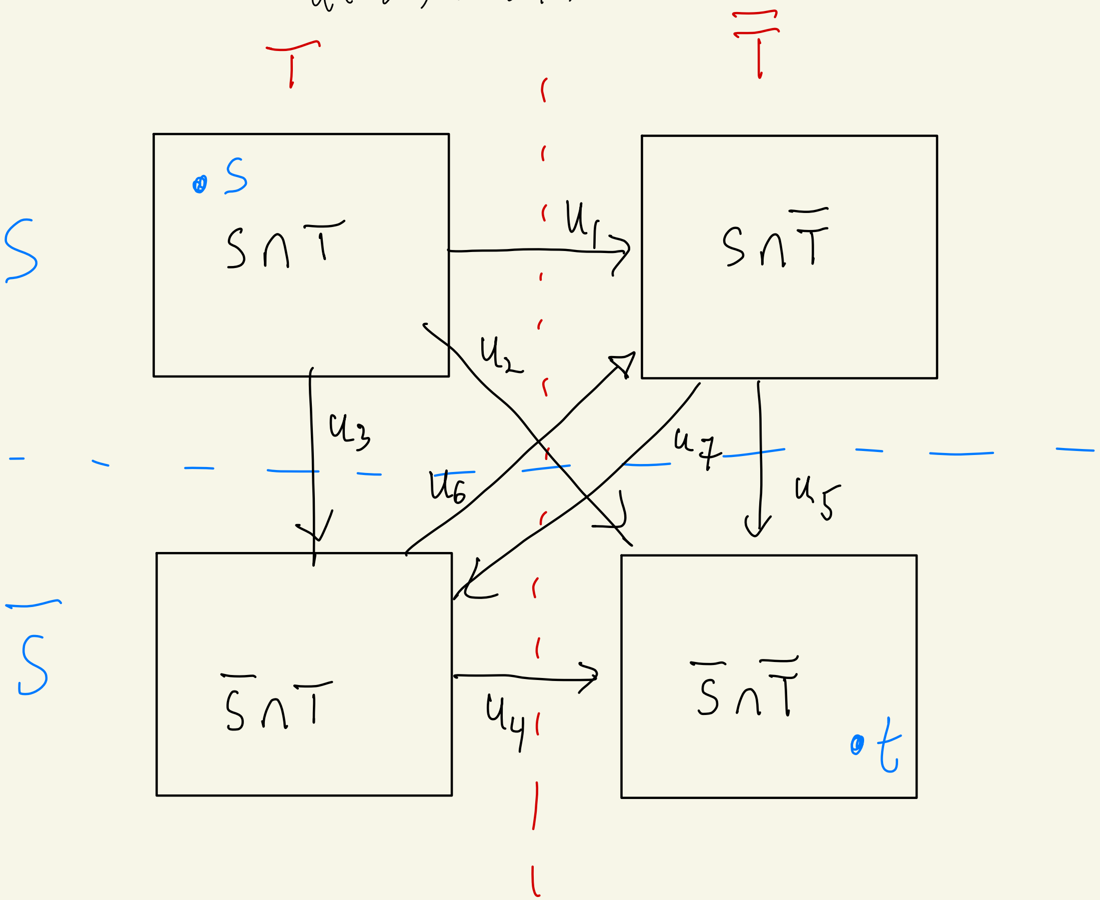
BJG 3.33

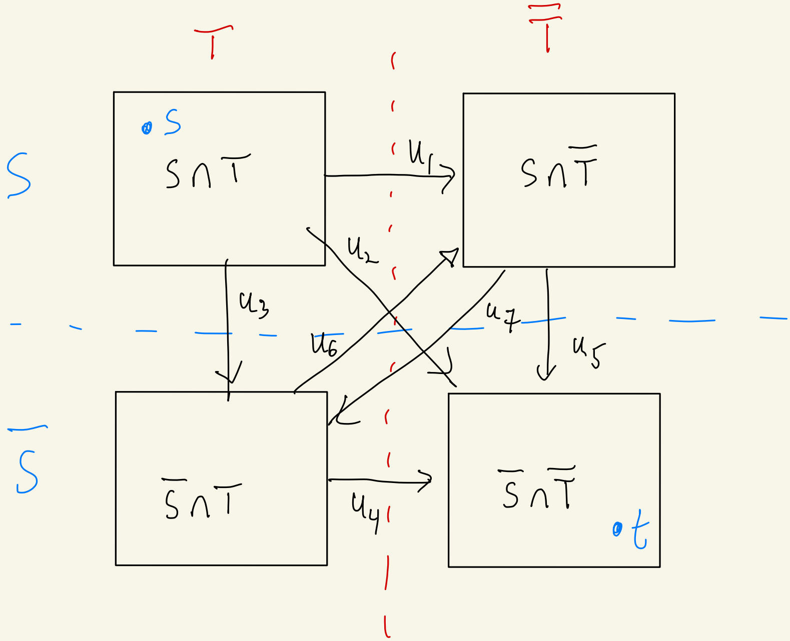
$N = (V \cup \{s,t\}, A, l \equiv 0, u)$

claim: $(S,\bar{S})$ and $(T,\bar{T})$ (s,t)-cuts

$\Downarrow$

$u(S,\bar{S}) + u(T,\bar{T}) \geq u(S\cap T, \overline{S\cap T}) + u(S\cup T, \overline{S\cup T})$





$$u(S\cap T, \overline{S\cap T}) + u(S\cup T, \overline{S\cup T})$$

$$= (u_1 + u_2 + u_3) + (u_2 + u_4 + u_5)$$

$$\leq (u_2 + u_3 + u_5 + u_7) + (u_1 + u_2 + u_4 + u_6)$$

$$= u(S, \bar{S}) + u(T, \bar{T})$$

BJG 3.34

If $(S, \bar{S})$ and $(T, \bar{T})$ are minimum (s,t)-cuts then $(S \cap T, \overline{S \cap T})$ and $(S \cup T, \overline{S \cup T})$ are also minimum (s,t)-cuts

let $K = u(S, \bar{S}) = u(T, \bar{T})$

then

$$K + K = u(S, \bar{S}) + u(T, \bar{T})$$

by 3.33

$$\geq u(S \cap T, \overline{S \cap T}) + u(S \cup T, \overline{S \cup T})$$

$$\geq K + K$$

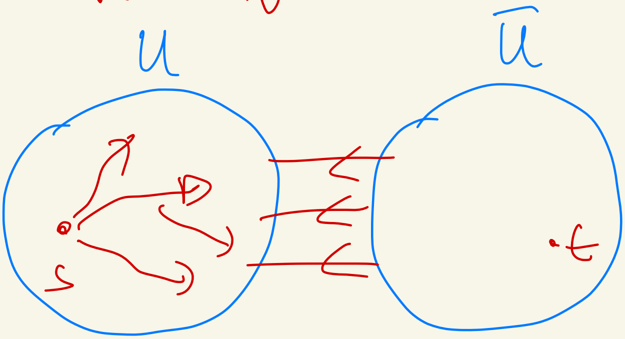
BJG 3.35

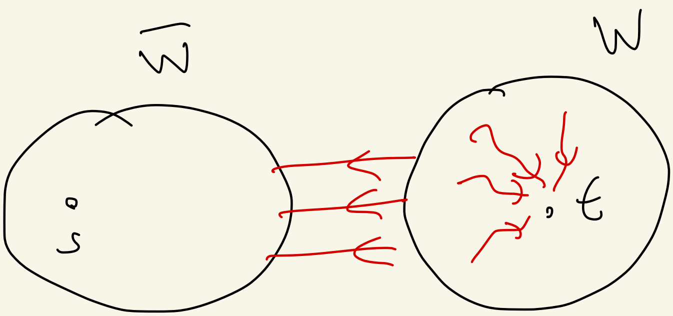
$N = (V \cup \{s,t\}, A, \ell = 0, w)$

Let x be a max flow in N

Let $U = \{i \mid \exists\, (s,i)\text{-path in } N(x)\}$

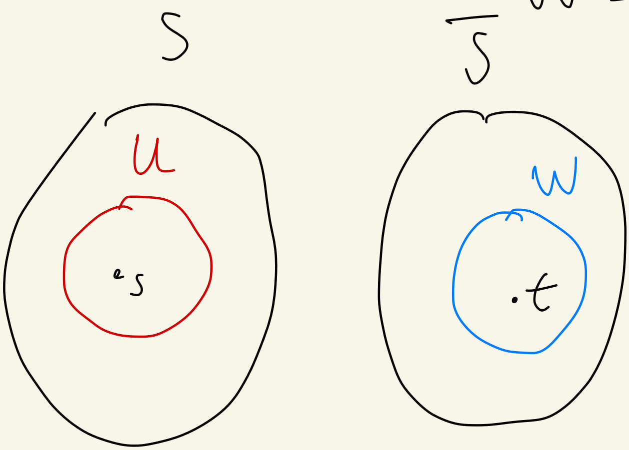
$W = \{j \mid \exists\, (j,t)\text{-path in } N(x)\}$





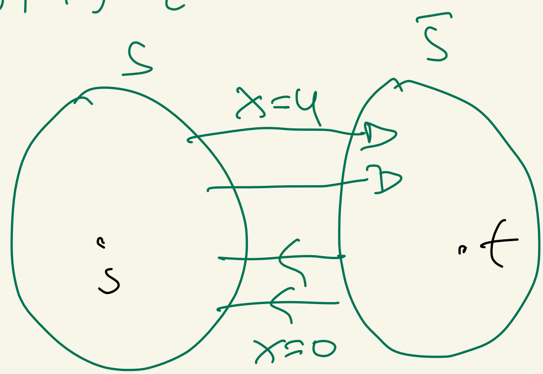
MFMC thm $\Rightarrow$ $(U, \bar{U})$ and $(\bar{W}, W)$ are min cuts

__Claim__ $\forall$ min cut $(S, \bar{S})$ we have $U \subseteq S$ and $W \subseteq \bar{S}$



__Note__ if x is a max flow and $(S, \bar{S})$ is a min cut then

$$|x| = x(S,\bar{S}) - x(\bar{S},S)$$

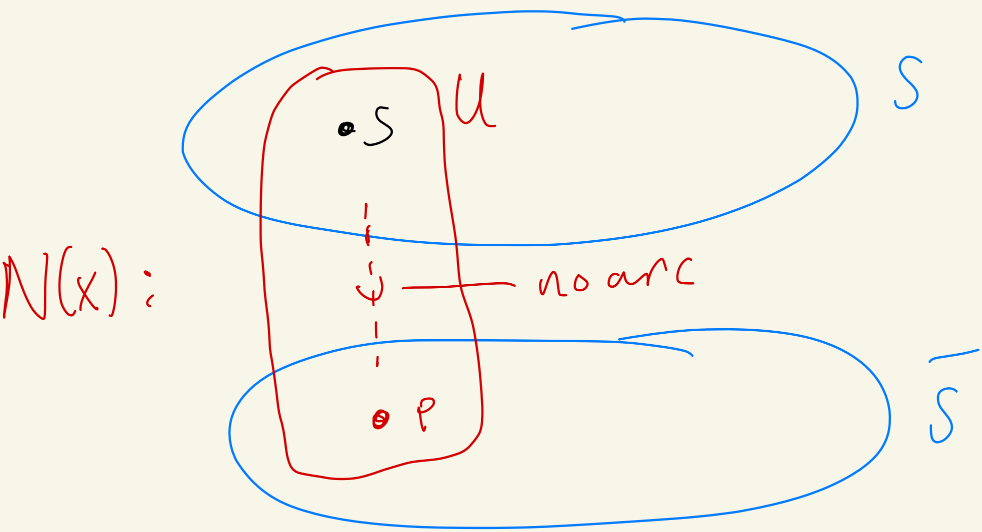


Sufficient to show that

$U \subseteq S$

Suppose $U \nsubseteq S$ then $U \cap \bar{S} \neq \emptyset$

($U \cap S \neq \emptyset$ as $x \in U \cap S$)



BJG 3.45

scheduling jobs on identical machines

- n jobs and a number of identical machines
- jobs has fixed schedule $[s_i, f_i]$

start time   finish time

- If a machine has processed job i then it needs t_{ij} processing time before it can process job j

$\boxed{\text{Goal: find a schedule that minimizes the number of machines we use}}$

$[s_i, f_i)$ $\qquad\qquad$ $[s_j, f_j)$

$$i \longrightarrow j$$

$$ij \in A \iff f_i + t_{ij} \leq s_j$$

minimizing # machines

$\Updownarrow$

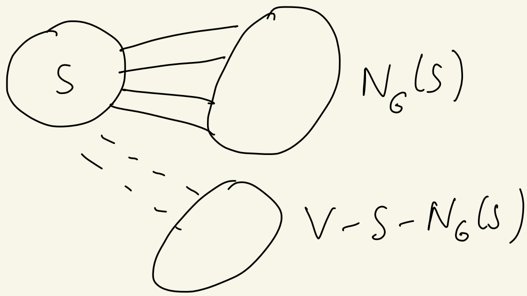
minimizing # paths covering all vertices in D (acyclic)

$\Updownarrow$

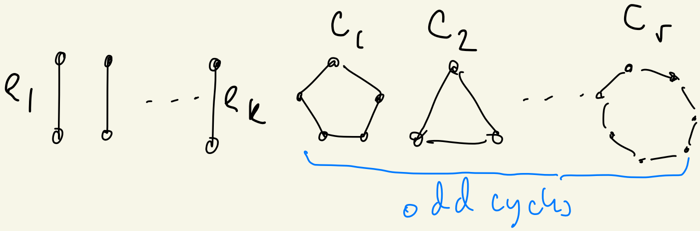
Finding min value integer flow in associated network.

BJG 3.55

$G = (V, E)$



Theorem Either G has

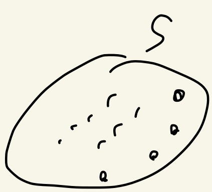


$(*)$

covering V or

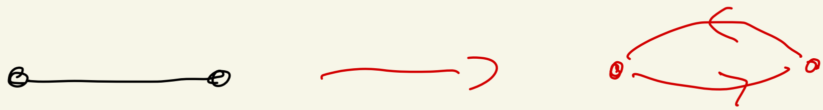
$\exists S \subseteq V$ s.t. S is independent and $|N_G(S)| < |S|$

If S exists then no $(*)$:



each e_i or C_j contributes a neighbour of S

Step 1: convert G to a digraph D



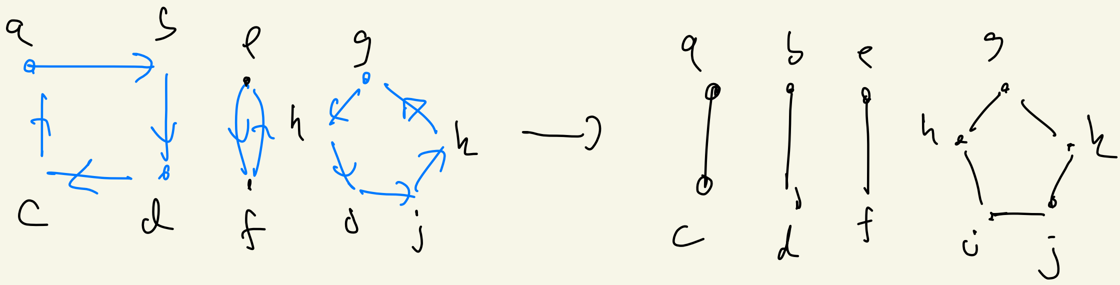
Claim A:

G has a collection as in $(*)$

$\Updownarrow$

D has a collection of disjoint cycles that cover $V(D) = V$

cycle factor



Claim B $\quad D$ has no cycle factor

$\Updownarrow$

$\exists S \subseteq V(D)$ s.t $|N^+(S)| < |S|$

[$N^+(S)$ is the set of out-neighbours of S in D]

Claim A + B $\Rightarrow$ Theorem :

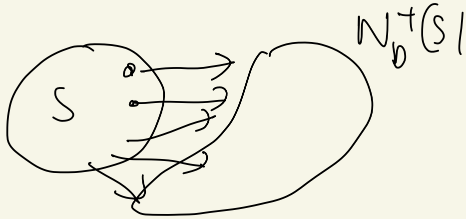
G has good collection (*)

$\Updownarrow$ claim A

D has a cycle factor

$\Updownarrow$ claim B

$\forall S$ independent $|N_D^+(S)| \geq |S|$

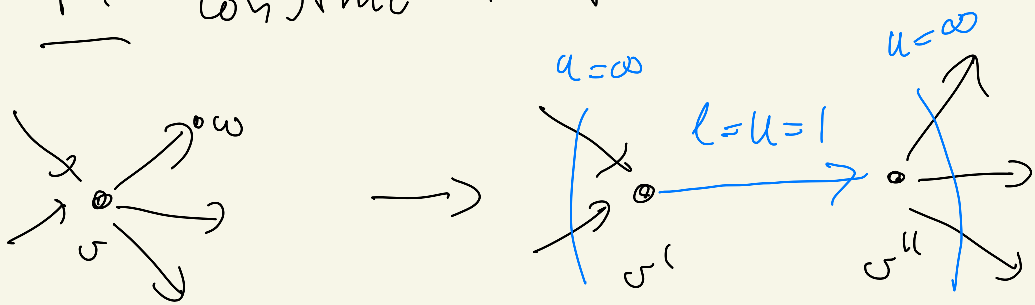


$\Updownarrow$

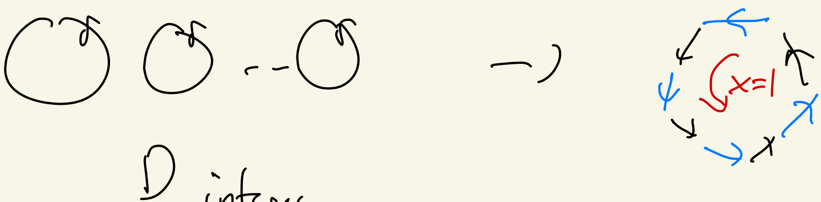
$\forall S$ indepent in G $|N_G(S)| \geq |S|$

Claim B D has no cycle factor $\iff \exists\, S \subseteq V(D)$ s.t. $|N^+(S)| < |S|$

Pf: construct N from D:



Note: N has a feasible circulation $\iff$ D has a cycle factor



x integer feasible circulation $\to$ x decomposes into cycle flows

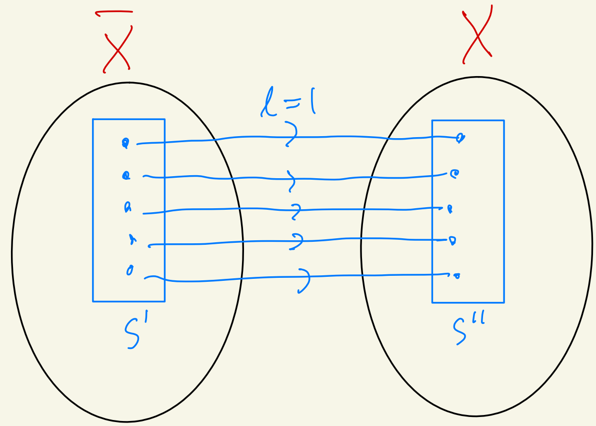
$\Downarrow$

disjoint cycles in D

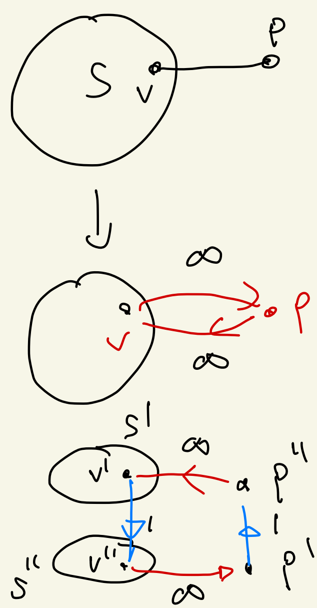
N has no feasible circulation

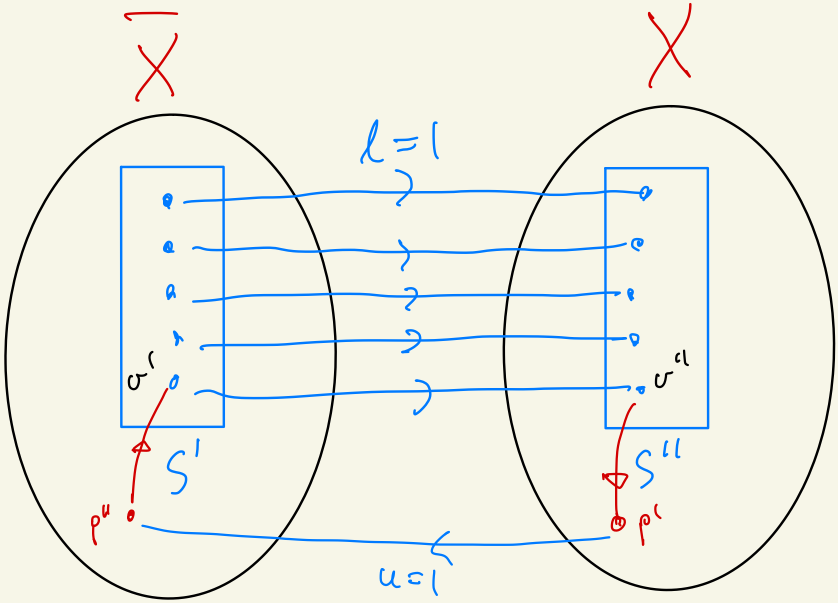
$\Updownarrow$ Hoffman's thm

$\exists X$ s.t. $\ell(\bar{X}, X) > u(X, \bar{X})$



- No arc from S'' to S' $\Rightarrow$ S is independent
- $|N_G(S)| < |S|$





each unique neighbour p of S gives an arc of capacity 1 from X to $\bar{X}$

$$|S| = \ell(\bar{X}, X) > u(X, \bar{X}) \geq |N_G(S)|$$