BJG 3.56

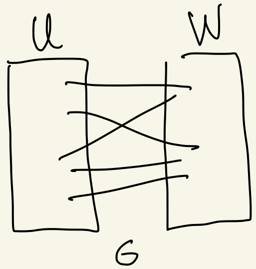
Theorem (Kőnig's)

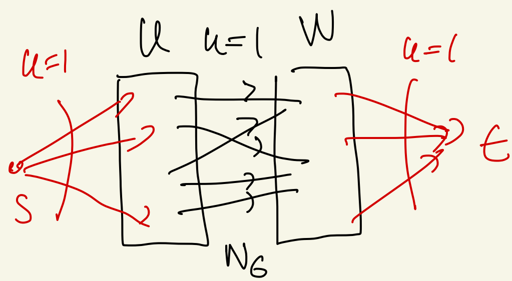
Every regular bipartite graph has a perfect matching

P: (via integrality theorem for max flows)

Let $G=(U,W,E)$ be k-regular, that is, $d(v)=k \ \forall v \in V(G)$

Then $k \cdot |U| = k \cdot |W|$ so $|U|=|W|$





The flow x defined by

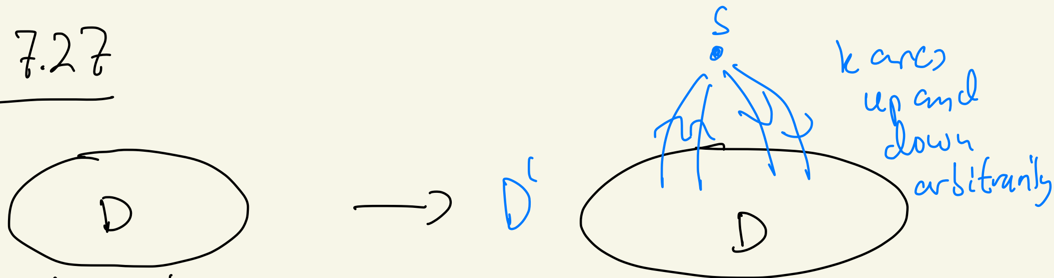
$x_{sv}=1 \ \forall v \in U$ $x_{wt}=1 \ \forall w \in W$ $x_{ij}=\frac{1}{k} \ \forall ij \in E(G)$

is feasible in N_G and has value $|U|=|W|$

By the integrality theorem: N_G has a feasible integer valued flow y of value $|U|$. Thus

$M=\{ij \in E \mid y_{ij}=1\}$ is a perfect matching

BJG 7.27



$\lambda(D) \geq k$

Claim: $\lambda(D') = k$

P: suppose $d^+_{D'}(X) < k$ for some $X \subseteq V(D')$

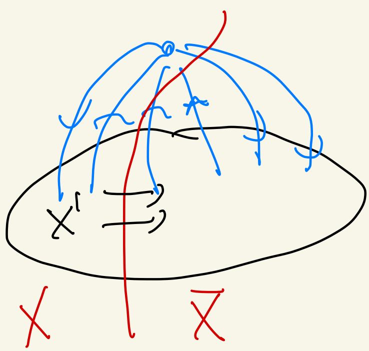
We can assume that $s \in X$ since otherwise we look at $\bar{X} = V \cup \{s\} \setminus X$ and $d^-(\bar{X}) < k$

cannot have $X = \{s\}$ as $d^+(s) = k$

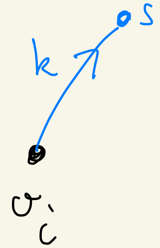
so $X' = X - s$ is non empty and

$d^+_D(X') \leq d^+_{D'}(X) < k$ } as $\lambda(D) \geq k$

[$d^+_D(X') = d_{D'}(X) - d_{D'}(s, V - X')$]



BJG 7.28



determine how many arcs from v_i to s we can delete and still have

(□) $d^+(U) \geq k \ \forall \emptyset \neq U \subset V$

- Only sets containing v_i are affected when we delete arcs $v_i \to s$.
- If $d^+(U) = k+r$ before deletion, then we can delete at most r arcs $v_i \to s$
- So we seek $g = \min\{d^+(U) \mid v_i \in U, U \subset V\}$
- By Mengers theorem, for a fixed $t \in V - v_i$, $\min\{d^+(U) \mid v_i \in U, t \notin U\}$ is precisely $\lambda(v_i, t)$ which is the maximum # of arc-disjoint (v_i, t)-paths in D
- $\lambda(v_i, t)$ can be found by one max flow calculation in N_D which is D with $u_{v_a v_b} = 1$ for all arcs $v_a v_b$ and

  $u_{v_a s}$ = current # of arcs to s

Hence

$$g = \min\{d^+(U) \mid v_i \in U \subset V\}$$

$$= \min\{\lambda(v_i, t) \mid t \in V - v_i\}$$ (n-1) maxflow calculations

and we can delete

$r = \min\{k, g-k\}$ arcs from v_i to s

and still satisfy

$$d^+(U) \geq k, \quad \forall \emptyset \neq U \subset V$$

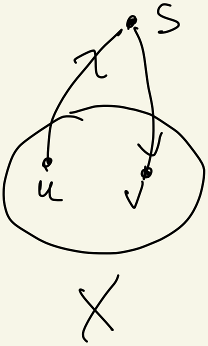
and

if $r < k$ then there exists a set $U \subset V$ with $v_i \in U$ s.t $d^+(U) = k$ after we delete r arcs from v_i to s

In total to find i minimal #tot arcs from V to s we need $n \cdot (n-1)$ maxflow calculations each of which take $O(n^{2/3} m)$ (why?)

BJG 7.30

We know: (us, su) is an admissible splitting

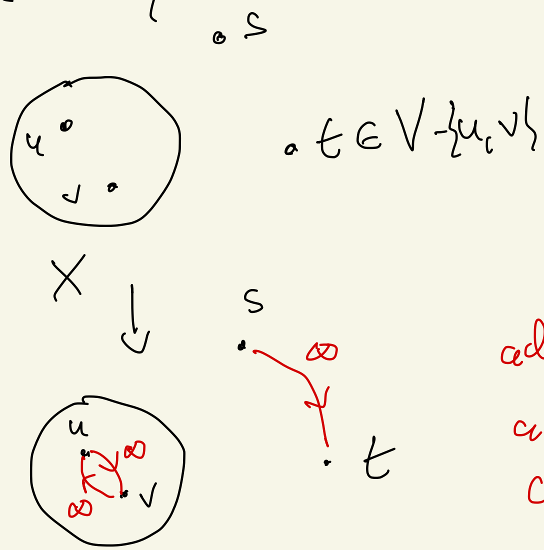


$\Updownarrow$

$\forall X \subset V$ s.t. $u, v \in X$ we have

$d^+(X), d^-(X) \geq k+1$

How do we find $\min\{d^+(X) \mid u, v \in X \subset V\}$
(and $\min\{d^-(X) \mid u, v \in X \subset V\}$?



add red arcs of cap ∞

all other arcs

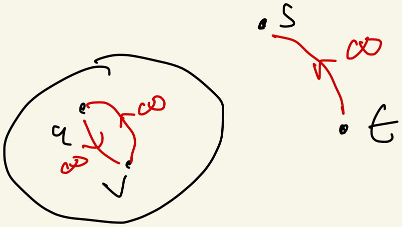
cap 1

Find max flows X_{ut} for all $t \in V - \{u, v\}$

If $\min\{|x_{ut}| \mid t \in V - \{u,v\}\} = k$

then (us, sv) is not admissible.

otherwise find $\min\{d^-(X) \mid u,v \in X \subset V\}$



Find max flows y_{tu} for all $t \in V - \{u,v\}$

If $\min\{|y_{tu}| \mid t \in V - \{u,v\}\} = k$

then (us, sv) is not admissible

otherwise (us, sv) is admissible.

Ahuja 10.6 $N = (V, A, \ell \equiv 0, u, b, c)$

Let x and x' be distinct feasible flows in N

- Then $\exists$ circulation $\tilde{x}$ in $N(x)$ such that

  $x' = x \oplus \tilde{x}$

- $\tilde{x}$ decomposes into cycle flows

  $W_1, W_2, \dots, W_p$ for some $p \leq m$

- All these cycles have positive flow on all their arcs in $N(x)$ (W_i has at least $\delta(W_i)$)

- Thus $\forall i \in [p]$ the cycle $\overleftarrow{W_i}$ (W_i reversed) is a cycle in $N(x')$

- Let $\bar{x}$ be the flow in $N(x')$ which sends $\delta(W_i)$ units along $\overleftarrow{W_i}$ for each $i \in [p]$.

- Then $x = x' \oplus \bar{x}$

Aufgabe 10.9 $N=(V,A,\ell\equiv 0,u,b,c)$

Let x' be feasible in N and let x be a pseudoflow ($b_x \neq b$) in N

- Then exist a flow $\tilde{x}$ in $N(x)$ s.t. $x' = x \oplus \tilde{x}$

- $\tilde{x}$ decomposes into some path flow along paths $P_1, P_2 \dots P_r$ and some cycle flows

- Each P_i starts in a vertex v with $b_x(v) < b(v)$ and ends in a vertex w with $b_x(w) > b(w)$

- For each such P_i the reverse path $\overleftarrow{P_i}$ (from w to v) is in $N(x')$ as we send $\delta(P_i) > 0$ units along P_i in $N(x)$

Ahuja 10.25 constrained max flow problem

maximize v

s.t $b_x(s) = v = -b_x(t)$, $b_x(i) = 0 \ \forall i \neq s,t$

(□) $0 \le x_{ij} \le u_{ij}$

$$\sum_{ij \in A} c_{ij} x_{ij} \le D$$

Normal max flow problem + a budget

Assumption: $c_{ij} \ge 0 \ \forall ij \in A$ and no (s,t)-path of cost 0

a) let $v^* \in \mathbb{Z}$ and let x^* be optimal (s,t)-flow of value v^* in $N = (V \cup \{s,t\}, A, \ell \equiv 0, u, c)$ and let $z^* = cx^*$. let π be an optimal potential ($c^\pi_{ij} \ge 0 \ \forall ij \in N(x)$)

Then x^* is a solution to (□) with budget $D = z^*$:

suppose $\hat{x}$ is feasible in N, has cost at most D and has a higher value than v^*. Then $\exists \tilde{x} \in N(x^*)$ s.t $\hat{x} = x^* \oplus \tilde{x}$ and $b_{\tilde{x}}(s) = b_{\hat{x}}(s) - b_{x^*}(s) > 0$

Thus $c\tilde{x} + cx^* = c\hat{x} \le D$

$\Downarrow$

$c\tilde{x} + D \le D \Rightarrow c\tilde{x} \le 0$

$N(x)$ has no negative cycle as x^* is optimal

since $b_{\tilde{x}}(s) > 0$ the decomposition of $\tilde{x}$ contains at least one (s,t)-path and each such path must have

Ahuja 14.4

Given $N=(V\cup\{s,t\}, A, \ell\equiv 0, u)$ s.t

$b_{x^*}(s) < v^0$ when x^* is a maxflow

- Price for increasing u_{ij} to $u_{ij}+1$ is α_{ij}
- Goal: Find cheapest way to increase some capacities s.t new network $N'=(V\cup\{s,t\}, A, \ell\equiv 0, u')$ has an (s,t)-flow x of value $b_x(s)\ge v^0$
- let $c_{ij}(x_{ij}) = \begin{cases} 0 & \text{if } x_{ij}\le u_{ij} \\ (x_{ij}-u_{ij})\cdot\alpha_{ij} & \text{if } x_{ij}>u_{ij} \end{cases}$

- solve

$$\min \sum_{ij\in A} c_{ij}(x_{ij})$$

s.t

$$b_x(i) = \begin{cases} v^0 & i=s \\ 0 & i\ne s,t \\ -v^0 & i=t \end{cases}$$

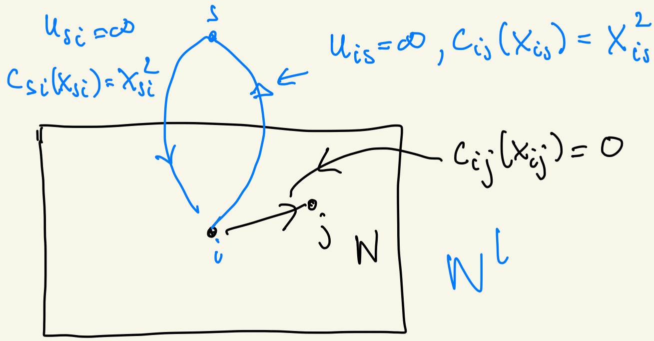
$$0 \le x_{ij} \le u'_{ij}$$

when u'_{ij} 'large enough' e.g. $u'_{ij}=v^0$

(if $u'_{ij}\ge v^0$ $\forall ij$ then we can send v^0 units from s to t)

Ahuja 14.5

Given $N=(V, A, l\equiv 0, u, b)$ with no feasible flow we wish to find a flow x with $0 \le x_{ij} \le u_{ij}$ $\forall ij \in A$ which minimizes $\sum_{i \in V} (b(i) - b_x(i))^2$



claim $k = \min \sum_{ij} c_{ij}(x_{ij})$

s.t. $b_x \equiv b'$

$0 \le x_{ij} \le u_{ij}$

$b'(s) = 0$

$b'(i) = b(i)$ $i \in V$

Solves the problem

claim $K = \min \sum_{ij} c_{ij}(x_{ij})$

s.t $b_x = b$

$0 \le x_{ij} \le u_{ij}$

solves the problem

- N has a feasible solution $\iff K = 0$

- Every feasible flow x' in N' corresponds to a pseudoflow x in N with $b(i) - b_x(i) = x'_{si}$ when $b_x(i) < b(i)$

and $b_x(j) - b(j) = x'_{sj}$ when $b_x(j) > b(j)$

and for this x' we have

$$\sum c_{ij}(x_{ij}) = \sum_{i \in V} \left(b(i) - b_x(i)\right)^2$$

So minimizing K solves our problem

Ahuja 14.14

x^* optimal solution to convex cost flow problem

$\Downarrow$ Exercise 14.15

$N(x^*)$ has no negative cycle.

- fix a vertex s and calculate shortest path distances $d(\,)$ from s ($O(n\log n + m)$ Dijkstra with Fib heaps)
- set $\pi = -d$ then $c^{\pi}_{ij} \geq 0 \;\forall ij \in N(x^*)$
- There is another optimal solution if and only if $N^{\pi}(x^*)$ has a cycle W s.t. all arcs of W have reduced cost $= 0$ (as $c(W) = c^{\pi}(W) \geq 0 \;\forall W$ as $c^{\pi}_{ij} \geq 0$)
- So we just need to check for a directed cycle in the subdigraph $D^0 = (V, A^0)$ where $A^0 = \{ij \mid c^{\pi}_{ij} = 0\}$
- This takes $O(n+m) = O(m)$ (assuming $m \geq n$) when π is already given if not then we need to find π ($= -d$) as above and then it takes time $O(n\log n + m)$

Ahuja 14.15

Claim x^* is optimal sol to convex cost flow prob

$\Updownarrow$ $N(x^*)$ (defined in section 14.4)

has no negative cycle.

$\Downarrow$: If W is negative cycle in $N(x^*)$

then $cx < cx^*$ when

$x = x^* \oplus \delta(W) \cdot W$ contradicting opt of x^*

$\Uparrow$: Assume $N(x^*)$ has no negative cycle and let x be feasible in N

(so $b_x \equiv b_{x^*}$)

Then $\exists$ cycles $W_1, W_2, \ldots, W_k$ in $N(x^*)$ such that $x = x^* \oplus \sum f(W_i) \cdot W$

Thus $c(x) - c(x^*) = C_{N(x^*)}\left(\sum_i \delta(W_i) \cdot W_i\right)$ cost in $N(x^*)$

$$c(x) - c(x^*) = \sum_{\{ij \mid x_{ij} > x_{ij}^*\}} \overset{(*)}{\left[c_{ij}(x_{ij}) - c_{ij}(x_{ij}^*)\right]} - \sum_{\{ij \mid x_{ij} < x_{ij}^*\}} \overset{(**)}{\left[c_{ij}(x_{ij}^*) - c_{ij}(x_{ij})\right]}$$

For each arc contributing to (*) $x_{ij} - x_{ij}^*$ is at least as large as $\delta(W_p)$ for all W_p s.t. $ij \in W_p$

so the contribution from the cost of W_p to such an arc ij is at most $\delta(W_p) \cdot \dfrac{c(x_{ij}) - c(x_{ij}^*)}{x_{ij} - x_{ij}^*}$

For an arc ij contributing to (**)
the difference $x_{ij}^* - x_{ij}$ is at least as large
as $\delta(W_q)$ for each W_q such that the arc $ji \in W_q$

so contribution from W_q to ij is at most

$$\delta(W_q) \cdot \frac{c(x_{ij}) - c(x_{ij}^*)}{x_{ij}^* - x_{ij}}$$

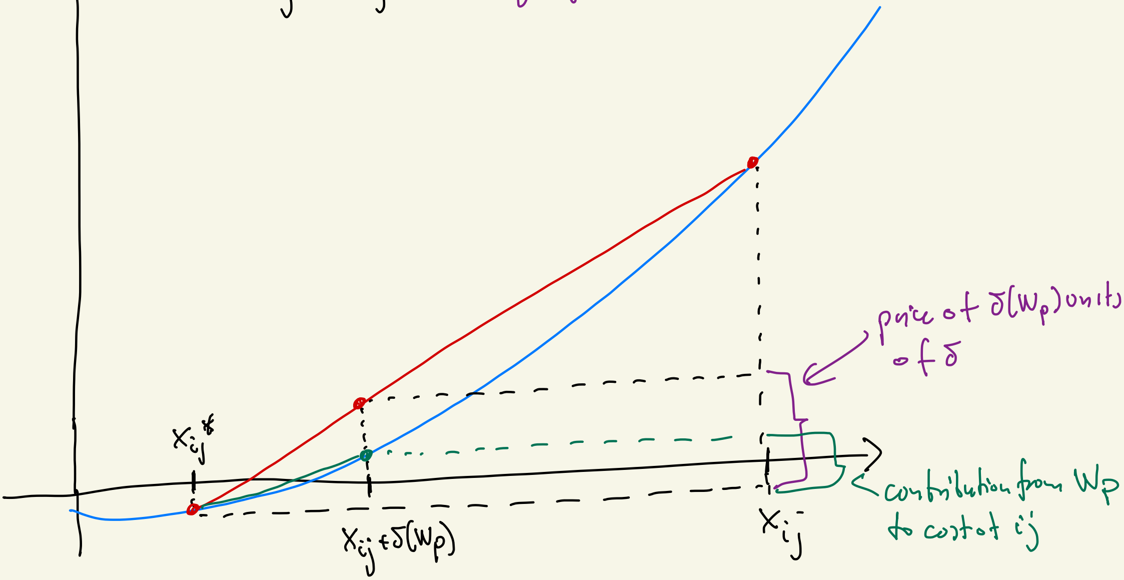
In both cases, the cost paid for ij on W_p or W_q is less than or equal to $\delta(W_p)$ $(\delta(W_q))$ units of the difference $|x_{ij} - x_{ij}^*|$

Conclusion

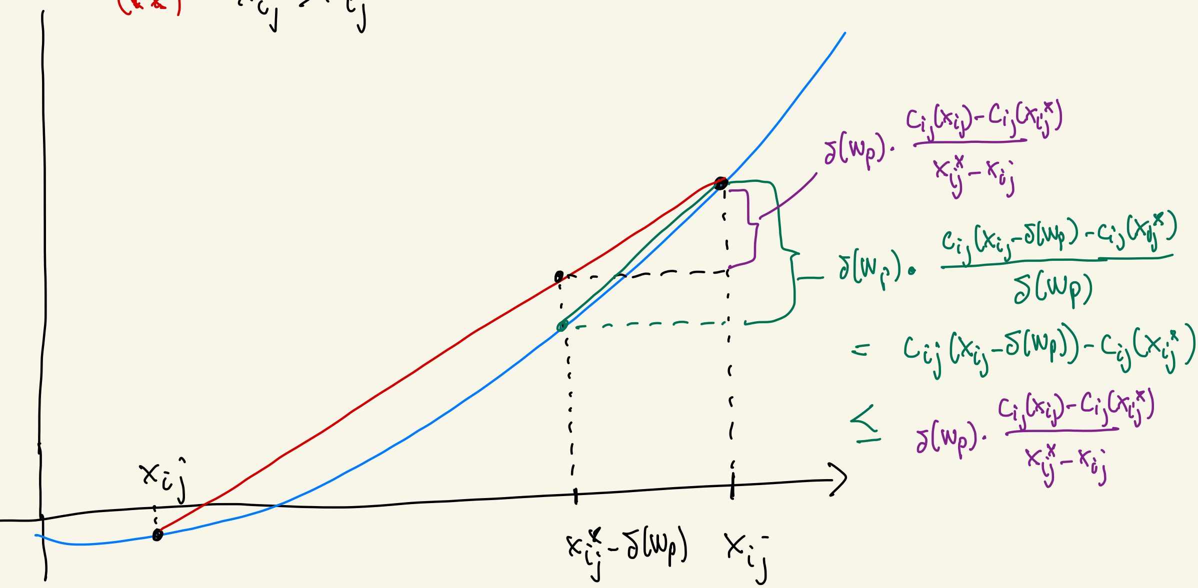
$$c(x) - c(x^*) \geq \sum \delta(W_p)\, c(W_p)$$

$$\geq 0 \quad \text{as } N(x^*) \text{ has no negative cycle.}$$

(*) $x_{ij} > x_{ij}^*$ $\delta = x_{ij} - x_{ij}^*$



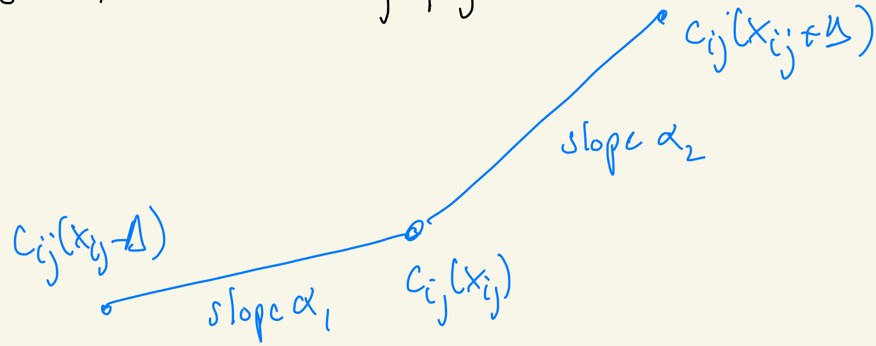
(**) $x_{ij}^* > x_{ij}$



Ahuja 14.17 capacity scaling algorithm

claim if $c_{ij}^{\pi} \geq 0 \ \forall ij \in N(x, 2\Delta)$

then one of $c_{ij}^{\pi}, c_{ji}^{\pi}$ must be non-negative in $N(x,\Delta)$



suppose ↓

$$c_{ij}^{\pi} = \frac{c_{ij}(x_{ij}+\Delta) - c_{ij}(x_{ij})}{\Delta} - \pi(i) + \pi(j) < 0$$

$$c_{ji}^{\pi} = \frac{c_{ij}(x_{ij}-\Delta) - c_{ij}(x_{ij})}{\Delta} - \pi(j) + \pi(i) < 0$$

⇓

$$\Delta\left(c_{ij}^{\pi} + c_{ji}^{\pi}\right) = c_{ij}(x_{ij}+\Delta) + c_{ij}(x_{ij}-\Delta) - 2c_{ij}(x_{ij}) < 0$$

⇓ $c_{ij}(x_{ij}) + \Delta\alpha_2 + c_{ij}(x_{ij}) - \Delta\alpha_1 - 2c_{ij}(x_{ij}) < 0$

⇓ $\Delta(\alpha_2 - \alpha_1) < 0$

impossible as $c_{ij}(\,)$ is a convex function

so $\alpha_2 \geq \alpha_1$