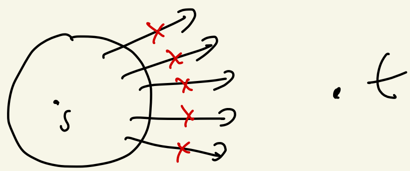
Ahuja 19.3

$G = (V, A)$ digraph with special vertices s, t

cost c_{ij} of destroying arc ij

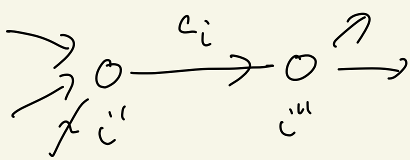
a) find min cost set of arcs that destroys all (s,t)-paths



Let $N = (V, A, \ell \equiv 0, u \equiv c)$

Claim min capacity of (s,t)-cut in N = min cost of a destroying set

b) May also kill vertices, killing vertex i costs c_i. Can handle this via vertex splitting



N'

Claim min cap of (s,t)-cut in N' = min cost of a destroying set of vertices & arcs

8.53 Thm 8.9.1 $M=(V,A,E)$

mixed graph $G=(V,E)$ undirected part

$D=(V,A)$ directed part

$$k \leq \min\{\tfrac{1}{2} d_G(X) + d^-_D(X) \mid \emptyset \neq X \neq V\}$$

Then the edges of G can be oriented as a digraph $D'=(V,A')$ such that $\hat{D}=(V,A\cup A')$ is k-arc-strong

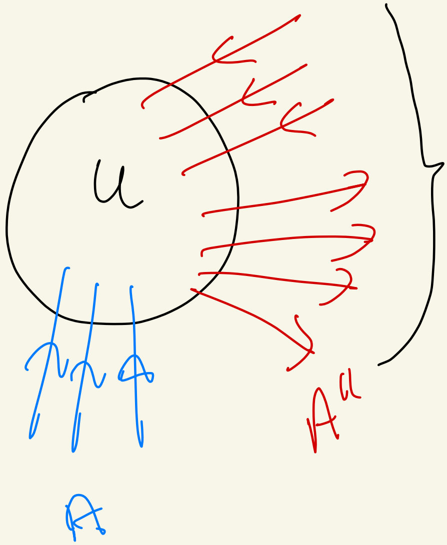
P: Let $D''=(V,A'')$ be any orientation of G (the undirected part of M) as usual $x: A'' \to \{0,1\}$ will follow which arcs to reverse

$$d^-_D(U) + d^-_{D''}(U) + x^+(U) - x^-(U) \geq k \quad (*)$$

must hold for all $U \neq \emptyset, V$

We claim that $x \equiv \frac{1}{2}$ satisfies $(*)$

and then, by the integrality theorem for submodular flows, there exists a good orientation A' of E such that $\hat{D}=(V,A\cup A')$ is k-arc-strong



x has values on these arcs

$$d^-_D(U) + d^-_{D''}(U) + x^+(U) - x^-(U) \qquad x = \tfrac{1}{2}$$

$$= d^-_D(U) + d^-_{D''}(U) + \tfrac{1}{2} d^+_{D''}(U) - \tfrac{1}{2} d^-_{D''}(U)$$

$$= d^-_D(U) + \tfrac{1}{2}\left(d^+_{D''}(U) + d^-_{D''}(U)\right)$$

$$= d^-_D(U) + \tfrac{1}{2} d_G(U)$$

$$\geq k$$ by the assumption in the theorem

BJG 8.64

$G=(V,E)$ for each edge ij we have two costs c_{ij} and c_{ji} where c_{ij} is the cost of orienting the edge ij as $i \to j$ and c_{ji} is the cost of orienting it as $j \to i$

Goal: find a k-arc-strong orientation of G of minimum cost (assume $\lambda(G) \geq 2k$ so there is a k-arc-strong orientation)

Let D be the orientation which orients each arc $ij \in E$ as $i \to j$ if $c_{ij} \leq c_{ji}$ and as $j \to i$ otherwise. Let C denote the cost of this orientation.

Let c' be the cost function which assigns cost $|c_{ij} - c_{ji}|$ to the arc between i and j in D

let $x(a)=1 \iff$ we reverse a

then we want

$$d^-_D(U) + x^+(U) - x^-(U) \ge k$$

$$\Updownarrow$$

$$(\square) \qquad \begin{aligned} x^-(U) - x^+(U) &\le d^-_D(U) - k \\ 0 \le x &\le 1 \end{aligned}$$

Every integer solution x to $(\square)$ corresponds to a k-arc-strong orientation D_x

and $c'x = c(D_x) - c(D)$

So every minimum cost integer valued submodular flow corresponds to an optimal k-arc-strong orientation of G.

Problem on WN 10

Show how to decide whether $G=(V,E)$ has k edge-disjoint spanning trees via submodular flows

Claim $G=(V,E)$ has k edge disjoint spanning trees (1)

$\Updownarrow$

$\forall r \in V$
$\exists$ orientation D of G which has k arc-disjoint out-branchings rooted at r (2)

$\Uparrow$: suppose $B^+_{r,1}, \dots, B^+_{r,k}$ are arc-disjoint out-branchings from r, let T_i be obtained from $B^+_{r,i}$ by ignoring the orientations. Then $T_1, T_2, \dots, T_k$ are edge disjoint spanning trees

$\Downarrow$ let $r \in V$ be given and let $\hat{T}_1, \tilde{T}_2, \dots, \hat{T}_k$ be edge-disj. sp. trees in G. orient T_i as an out-branching $\hat{B}^+_{r,i}$ from r. Then $\hat{B}^+_{r,1}, \hat{B}^+_{r,2}, \dots, \hat{B}^+_{r,k}$ are arc-disj out-branchings rooted at r. (II)

Let D' be an arbitrary orientation of G

We can obtain any other orientation D by reversing zero or more arcs of D'

We can use a flow $x : A \to \{0,1\}$ to show which arcs we reverse

$x(a) = 1 \iff a$ is reversed

By Edmonds branching theorem, such an orientation D of G has k arc-disj out-branchings from r if and only if

$$d^-_D(U) \geq k \quad \forall \emptyset \neq U \subseteq V - r \quad (\square)$$

Let D_x denote D' with those arcs where $x = 1$ reversed. Then

$$d^-_{D_x}(U) = d^-_{D'}(U) + x^+(U) - x^-(U)$$

Hence (□) holds with $D = D_x$ if and only if

$$d^-_{D'}(U) + x^+(U) - x^-(U) = d^-_{D_x}(U) \geq k$$

$\Downarrow$

$$x^-(U) - x^+(U) \leq d^-_{D'}(U) - k = b(U)$$

$b(U) = d^-_{D'}(U) - k$ is submodular as

$d^-_{D'}(\,)$ is submodular

<u>Conclusion:</u> G has an orientation with k arc-disj out-branchings rooted at r

$\Updownarrow$

The submodular flow problem

$$x^-(U) - x^+(U) \leq d^-_{D'}(U) - k$$

$$0 \leq x(a) \leq 1 \quad \forall a \in A(D')$$

has a feasible solution.

Exam 2018 B Problem 1

Given $D=(V,A)$ with $d^+_D(v)+d^-_D(v)$ even $\forall v\in V$

a) Find a set of arcs to reverse such that new digraph D' is eulerian ($d^+_{D'}(v)=d^-_{D'}(v)$ $\forall v\in V$)

as usual: x flow in $N=(V,A,\ell\equiv 0,u\equiv 1,b)$

to be determined

$x(a)=1 \Leftrightarrow$ reverse a

Then we want

$$d^-_D(i)+\sum_{ij\in A}x_{ij}-\sum_{ji\in A}x_{ji}=\frac{d^+_D(i)+d^-_D(i)}{2} \qquad \forall i\in V$$

$$\Updownarrow$$

$$b_x(i)=\frac{d^+_D(i)+d^-_D(i)}{2}-d^-_D(i)=b(i)$$

Every eulerian reorientation D' of D corresponds to a feasible integer flow x in

$$N=(V,A,\ell\equiv 0,u\equiv 1,b)$$

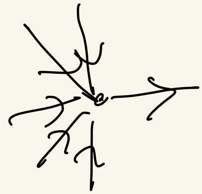
This is a unit capacity network so by Exam 20 A problem 4.3 such a flow can be found in time $O(n^{2/3}m)$

b) we can find the minimum # of arcs to reverse by giving each arc a cost of 1 then the cost of a feasible flow x in $N = (V, A, l \equiv 0, u \equiv 1, b, c \equiv 1)$ is exactly the number of arcs we reverse in D to get an eulerian reorientation.

c) Goal: find smallest k such that we can obtain an eulerian reorientation of D without reversing more than k arcs incident to any vertex i.

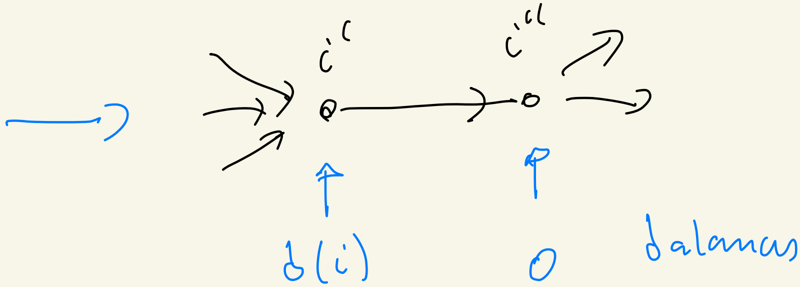
Observation: we must reverse at least

$$m(i) = \frac{|d^+_D(i) - d^-_D(i)|}{2}$$ arcs at i

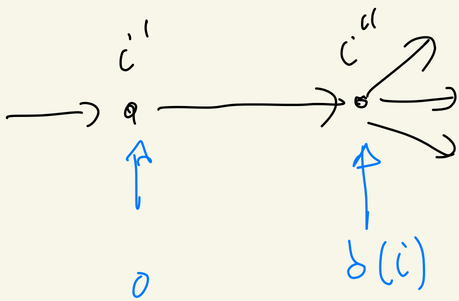


in order to reason how many more arcs we reverse at i, we perform vertex splitting and set balances as follows

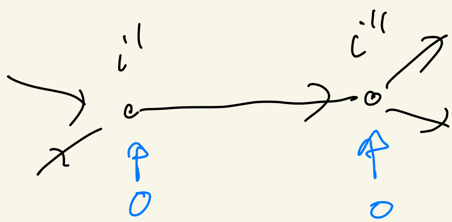
$d^-_D(i) > d^+_D(i)$



$d^-_D(i) < d^+_D(i)$



$d^-_D(i) = d^+_D(i)$



Example $\quad d^-_D(i) = 8 \quad d^+_D(i) = 2$

So $\quad m(i) = \frac{8-2}{2} = 3 \Rightarrow$ must reverse

at least 3 arcs at i and

$$b(i) = \frac{d^+_D(i) + d^-_D(i)}{2} - d^-_D(i) = 5 - 8 = -3$$

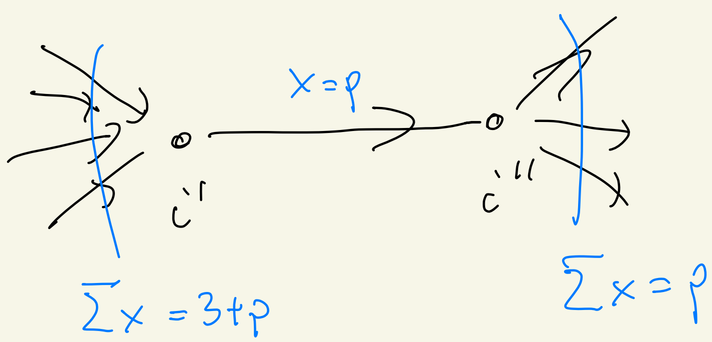
If we reverse more than 3 arcs at i

then we must reverse p more incoming and

p outgoing

In the new network this corresponds to

the following flow values at i', i''



Now we want to put a cost $C_{i'i''}(x_{i'i''})$
so that sending k units
on the arc $i' \to i''$ is more expensive
than sending $k-1$ units on
<u>all</u> arcs $j' \to j''$ $j \in V$

- Set $C_{i'i''}(x_{i'i''}) = m^{2x_{i'i''}+1}$
- $m^{2k+1} > m \cdot m^{2(k-1)+1} = m \cdot m^{2k-1} = m^{2k}$

so more expensive to send k units along $i' \to i''$
than $k-1$ units along all $j' \to j''$

$f(k) = m^{2k+1}$ is a convex function:

$f'(k) = (2k+1)m^{2k}$ and $f''(k) = 2k(2k+1)m^{2k-1}$

so $f'' \geq 0$ always $\Rightarrow f'(k)$ is increasing always

So we find a feasible flow x of min cost cost $c_{ij}(x_{ij}) = 0 \quad \forall ij \in A$

$$c_{i'i''}(x_{i'i''}) = m^{2x_{i'i''}+1}$$

$$\forall i \in V$$

Then is a feasible solution of cost less than m^{2k+1} if and only if we can find an eulerian reorientation by never reversing more than $2(k-1)$ arcs more than the minimum $m(i)$ at any vertex i

Problem 2, Exam 2016 B

a) Frank algorithm (very brief)

Let $D=(V,A)$ with $V=\{1,2,\dots,n\}$ and $k\in\mathbb{N}$ be given. Assume $\lambda(D)<k$

1. add a new vertex s to $D=(V,A)$, k arcs is $\forall i\in V$ and k arcs si $\forall i\in V$

Now we have $d^+_{D'}(U), d^-_{D'}(U)\geq k$ $\forall \emptyset\neq U\subset V(D)$ $\boxed{*}$ in resulting D'

2. For $i:=1$ to n
   remove as many is arcs as possible while preserving ($\boxed{*}$)

   let γ^- be the number of remaining arcs entering s

3. For $j:=1$ to n
   remove as many sj arcs as possible while preserving ($\boxed{*}$)

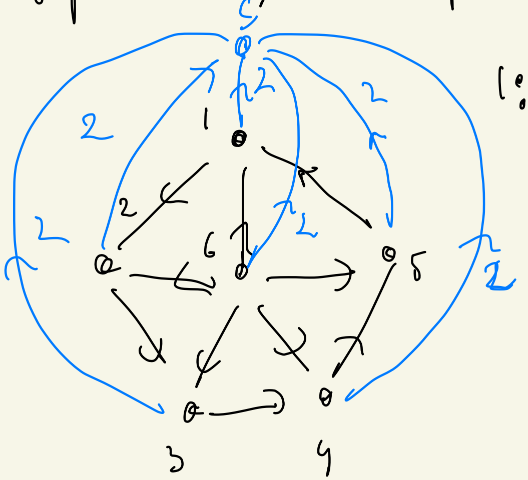
   let $\gamma^+ = \#$ arcs out of s

4. If $\gamma^-<\gamma^+$ add $\gamma^+-\gamma^-$ arcs $1\to s$

5. if $\gamma^->\gamma^+$ add $\gamma^--\gamma^+$ arcs $s\to 1$

6. while $d^+(s)>0$
   a. take arc sj b. find is such (is,sj) is admissible c. split is with sj

step 2 only arcs up shown drop subscript always current digraph



1: $d^+(1)=3$

$d^+(X) \geq 4 \quad \forall |X|=2$ with $1 \in X$

$\Downarrow$ can delete one arc $1 \to s$

2: $d^+(2)=3$, $d^+(X) \geq 4$ for all X, $|X| \geq 2$ $2 \in X$

$\Rightarrow$ delete one arc $2 \to s$

3: $d^+(3)=3$ $d^+(X) \geq 4$ for all X, $|X| \geq 2$ $3 \in X$

$\Rightarrow$ delete one arc $3 \to s$

4: $d^+(4)=3$, $d^+(X) \geq 4$ $\forall X$, $|X| \geq 2$ $4 \in X$

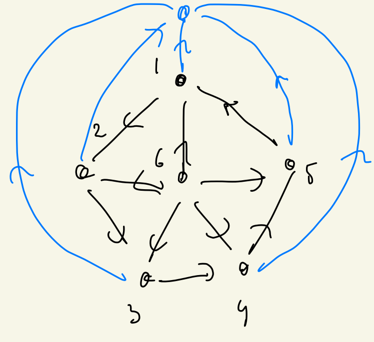
$\Rightarrow$ delete one arc $4 \to s$

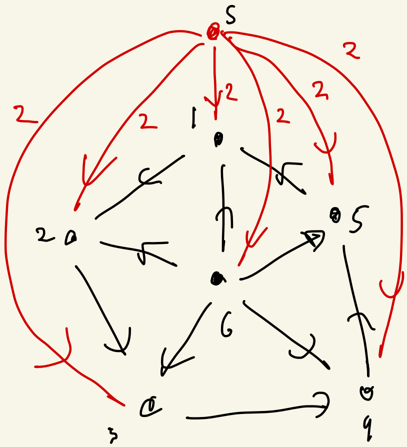
5: $d^+(5)=3$, $d^+(X) \geq 4$ $\forall X$, $|X| \geq 2$ $5 \in X$

$\Rightarrow$ delete one arc $5 \to s$

6: $d^+(X) \geq 4$ for all X containing 6

$\Rightarrow$ delete 2 arcs $6 \to s$





1: $d^-(X) \geq 4 \ \forall X \ni 1 \Rightarrow$ delete both arcs $s \to 1$

2: $d^-(X) \geq 4 \ \forall X \ni 2 \Rightarrow$ delete both arcs $s \to 2$

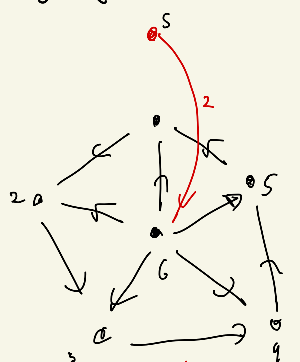
To see this: if $6 \in X$ then $d^-(X) \geq d(s,2) + d(s,6) \geq 4$ so may assume $6 \notin X$ if $1 \in X$ then $d^-(X) \geq d(s,2) + d(6,\{1,2\}) \geq 4$ and if $1 \notin X$ then $d^-(X) \geq d(s,2) + d(\{1,6\},2) \geq 4$

3: $d^-(X) \geq 4 \ \forall X \ni 3$ so delete both arcs $s \to 3$

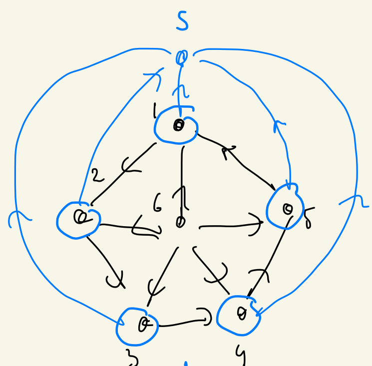
4: $d^-(X) \geq 4 \ \forall X \ni 4 \Rightarrow$ delete both arcs $s \to 4$

5: $d^-(X) \geq 4 \ \forall X \ni 5 \Rightarrow$ delete both arcs $s \to 5$

6: $d^-(6) = 0$ so keep both arcs $s \to 6$.



$\gamma^+ = 2$



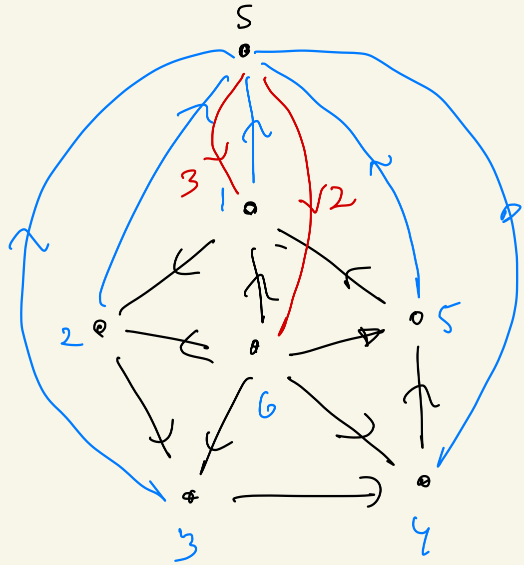
$\gamma^- = 5$

The subpartition X_1, X_2, X_3, X_4, X_5 with $X_i = \{i\}$ shows that we need 5 arcs

$\gamma^- - \gamma^+ = 3$

so we add 3

arcs $s \to 1$



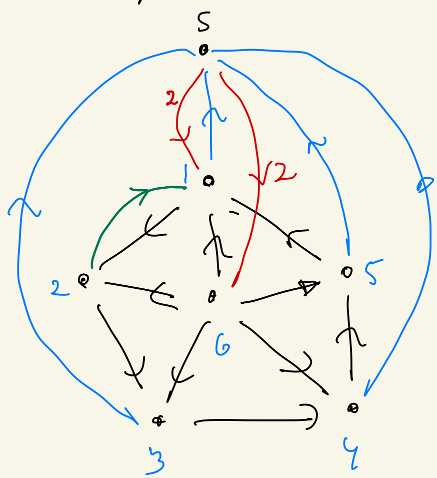
we first split off the 3 arcs $s \to 1$ and then the 2 arcs $s \to 6$

$(1s, s1)$ is not admissible as $d^+(1) = 2$

$(2s, s1)$: $d^-(X) \geq 2$ for all X containing $\{1,2\}$ due to red arcs

$d^+(X) \geq 2$ for all $X \supseteq \{1,2\}$ due to the arcs

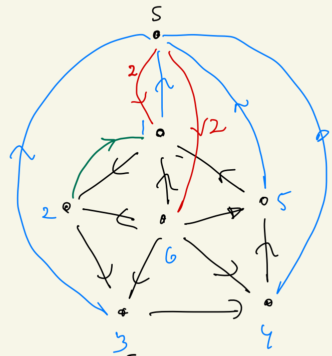
$\Rightarrow$ $(2s, s1)$ is admissible



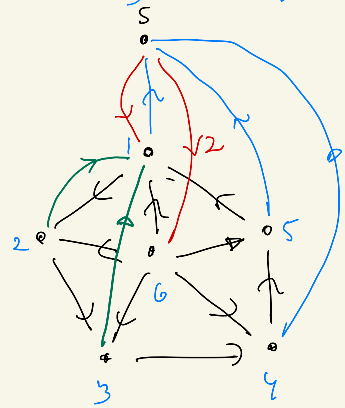
try $(3s, s1)$: $d^-(X) > 2$ for all X with $\{1,3\} \subseteq S$
due to red arcs and arcs out of 6

$d^+(X) > 2$ for all $X \supseteq \{1,3\}$
due to the cycle 123451
and blue arcs

so $(3s, s1)$ is admissible



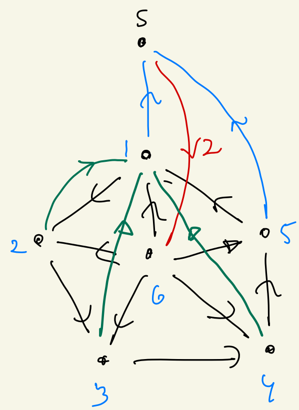
try $(4s, s1)$: $d^-(X) > 2$ $\forall X$ with $\{1,4\} \subseteq S$
$d^+(X) > 2$ $\forall X \supseteq \{1,4\}$ due to 123451
and blue arcs



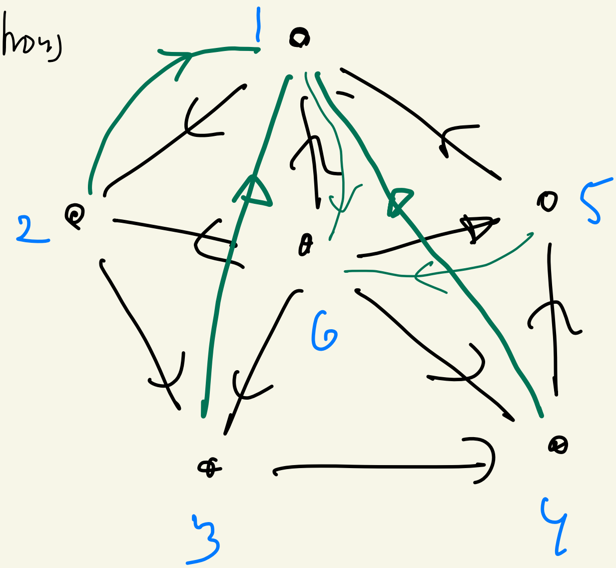
try $(1s, s6)$: $d^-(X) > 2$ $\forall X$ with $\{1,6\} \subseteq S$
$d^+(X) > 2$ $\forall X \supseteq \{1,6\}$ due
1s and 123451
so $(1s, s6)$ is admissible

and finally $(s5, s6)$ is admissible

trivially



resulting 2-arc-strong digraph



Subroutines:

A. Given $i \in V$ with k arcs to s:
how many can we delete

B. Is the splitting (is, sj) admissible.

A: is solved by calculating

$$r(i) = \min\{d^+(X) \mid i \in X, s \notin X, X \neq V\}$$

we can delete $g = \min\{k, r(i)-k\}$ arcs from i to s

For each $t \in V-i$: calculate (i,t)-flow $x^{(t)}$

in $D + s \xrightarrow{\infty} t$ and $u_{ij} = 1 \;\; \forall ij \in A$

Then $r(i) = \min\{|x^{(t)}| \mid t \in V-i\}$

by the max flow min cut thm.

B: is solved by calculation

$\alpha = \min\{d^+(X) \mid \{i,j\} \subseteq X \neq V\}$ and

$\beta = \min\{d^-(X) \mid \{i,j\} \in X \neq V\}$

For $t \in V - \{i,j\}$! calculate max (i,t)-flow

$y^{(i,t)}$ in $D + \; i \underset{\infty}{\overset{\infty}{\rightleftarrows}} j \quad \underset{s}{\circ} \xrightarrow{\infty} \underset{t}{\circ}$

and a max (t,i)-flow $z^{(t,i)}$ in

$D + \; i \underset{\infty}{\overset{\infty}{\rightleftarrows}} j \quad \underset{t}{\circ} \xrightarrow{\infty} \underset{s}{\circ}$

Then

$$\alpha = \min\{|y^{(i,t)}| \mid t \in V - \{i,j\}\}$$

$$\beta = \min\{|z^{(t,i)}| \mid t \in V - \{i,j\}\}$$

and (is, sj) is admissible precisely when $\alpha, \beta > k$

b) By Nash-Williams orientation theorem

$r^k(D) < \infty \iff UMG(D)$ is $2k$-edge-connected

c) $r^k(D)$ can be found by finding a min cost feasible submodular flow

see next page.

Interpret $x: A \to \{0,1\}$ in $N = (V, A, l \equiv 0, u \equiv 1)$ by reversing ij if and only if $x_{ij} = 1$

then resulting digraph D' has in-degree

$$d^-_{D'}(U) = d^-_D(U) + x^+(U) - x^-(U)$$

So we want an integer x s.t

$$d^-_D(U) + x^+(U) - x^-(U) \geq k$$

iff

$$x^-(U) - x^+(U) \leq d^-_D(U) - k = b(U)$$

and $0 \leq x \leq 1$

We saw in b) that $r^k(D) < \infty \iff \lambda(UMG(D)) \geq 2k$

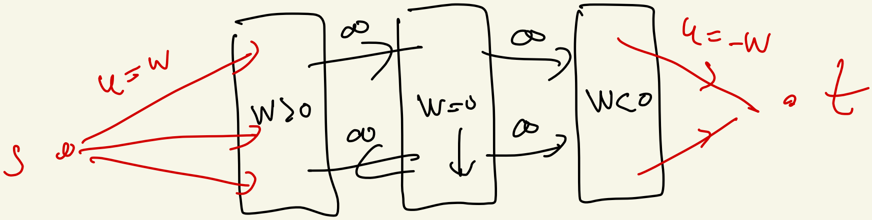
and By the Edmonds-Giles thm there is a feasible integer x whenever there is a feasible sol.

Set $c_{ij} = 1$ and finding a min cost feasible submodular flow x gives us $r^k(D)$ as the cost of such an x.

Problem 3 Exam 2018B

a) Given $D=(V,A,w)$ $w: V \to \mathbb{R}$

we make N as follows



all original arcs have ∞-cap.

We proved that if X is a closure then $(\overset{S}{X \cup \{s\}}, \overset{\bar{S}}{V \setminus X \cup \{t\}})$ is an (s,t)-cut

and $w(X)+u(S,\bar{S}) = \text{constant}$

so finding a min cut $(S,\bar{S})$ gives us a max closure $S-\{s\}$.

We can force v to be in our closure by setting $u_{sv}=\infty$ instead of $u_{sv}=w(v)$

We can force v' not to be in our closure by setting $u_{v't}=\infty$

Now we can find a maximum weight closure X with $X \neq \emptyset, V$ by running through all pairs of distinct vertices v, v' while forcing v to be in the closure and v' not to be in the closure.

$\binom{n}{2}$ max flow calculations so $O(n^5)$ with FIFO pfp

5) Goal find $\max\{|X| \mid d^{t}(X)=0, X \neq V\}$

Solution: set $w(v)=1 \; \forall v$ and find a max weight closure $X \neq \emptyset, V$ then this X gives max above.

Running time $O(n^5)$ as above

c) Suppose $d^+(X) = d^+(Y) = 0$ and $X \cup Y \neq V$. Then

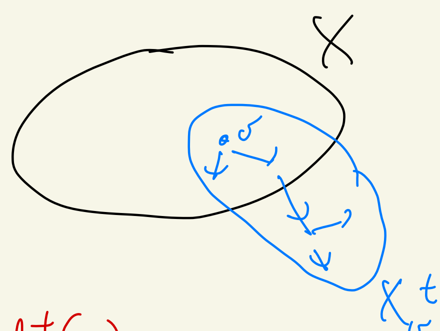
$$0 + 0 = d^+(X) + d^+(Y)$$
$$\geq d^+(X \cup Y) + d^+(X \cap Y)$$
$$\geq 0 + 0$$

so $d^+(X \cup Y) = 0$

d) $D = (V, A)$ $\quad X_v^+ = \{w \mid \exists\ (v,w)\text{-path in } D\}$

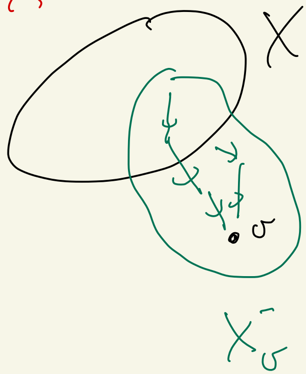
$X_v^- = \{z \mid \exists\ (z,v)\text{-path in } D\}$

If $d^+(X) = 0$ and $v \in X$ then $X_v^+ \subseteq X$ as otherwise there is an arc from $X \cap X_v^+$ to $X_v^+ \setminus X$ ⇝ $d^+(X) = 0$



if $d^+(X) = 0$ and $\sigma \notin X$

then $X \cap X_\sigma^- = \emptyset$ or we would have an arc from $X \cap X_\sigma^-$ to $V \setminus X$ implying $d^+(X) > 0$ $\lightning$



e) Quick algorithm for finding $\max \{ |X| \mid d^+(X) = 0, X \neq V \}$

Find X_σ^- forall σ in time (at most) $O(n(n+m))$

Note that $d^+(V \setminus X_\sigma^-) = 0$ so $V \setminus X_\sigma^-$ is a candidate for X

Since $X \neq V$ then is some $\sigma \in V \setminus X$ and $X_\sigma^- \cap X = \emptyset$ by d.

Conclusion

X is the complement of some X_v^- and this X_v^- has $|X_v^-| \leq |X_w^-|$ $\forall w \in V$

given $X_{v_1}^- \dots X_{v_n}^-$ we can find X in time $O(n)$

so $O(n(n+m))$ in total

i.e much faster than the flow based method in 6)

Problem 4 Exam 2018 B

a)

P: On input (D, c, r):

1. Check whether r can reach all other vertices and stop if No
2. For $v \in V-r$: $y_v \leftarrow \min\{c(uv) \mid uv \in A\}$
3. For $v \in V-r$: fix one arc a_v entering v with $c(a_v) = y_v$
4. Let $F^* = \{a_v \mid v \in V-r\}$
5. If F^* is a branching (no cycle) return F^*

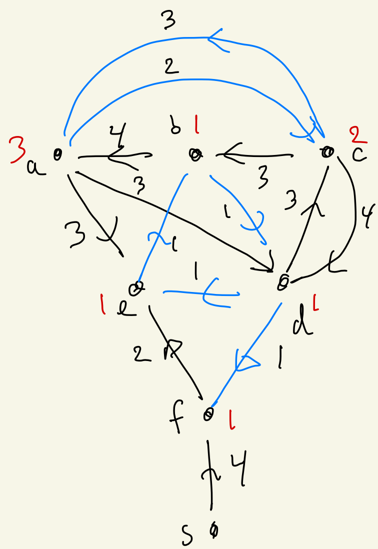
   Else let $C \subseteq F^*$ be a cycle

   a. $D \leftarrow D/C$

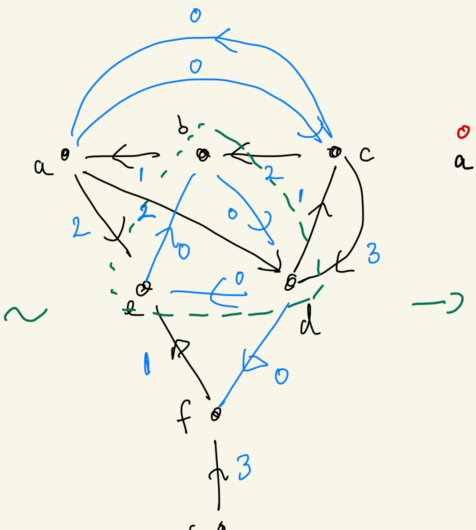
   b. $c' \leftarrow c - y$ $\quad (c'(uv) \leftarrow c(uv) - y_v \ \forall uv \in A)$

   c. Solve recursively on $(D/C, c', r)$

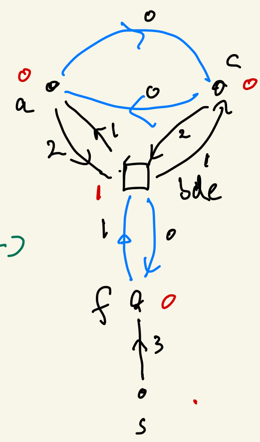
   d. Blow up C again and return the resulting branching



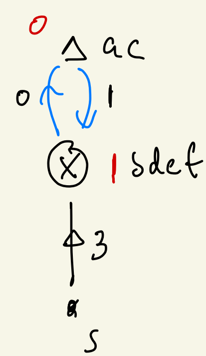
D, c, y $\quad \sum y = 9$



D, c'

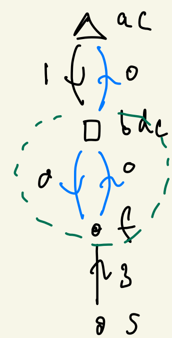


$D/bde, c^1, y$ $\quad \sum y = 1$

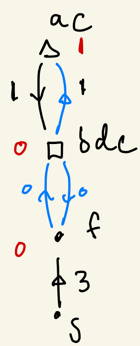


$D/bde/ac/bdef, c^3, y$

$\sum y = 1$

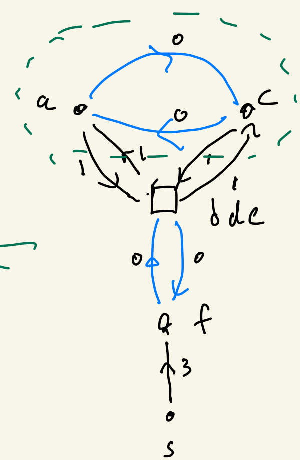


$D/bde/ac, c^3$

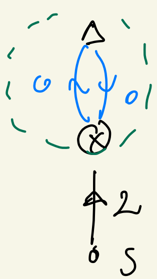


$D/bde/ac, c^2, y$

$\sum y = 1$



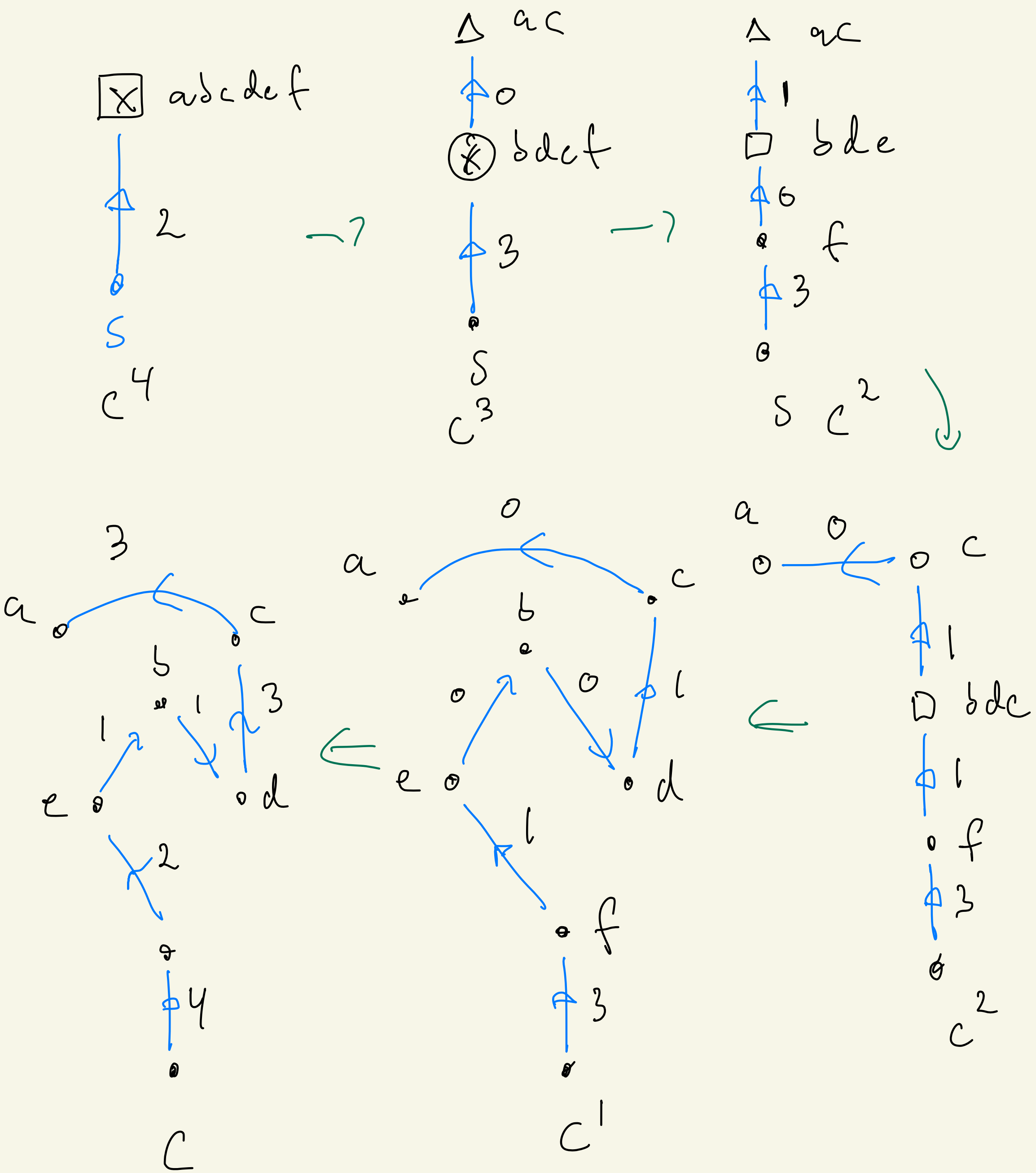
$D/bde, c^2$



c^4

$\sum y = 2$

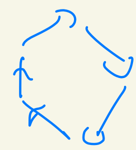
out-branching
so flow up the 3 contracted
cycles in reverse order



cost = 14 = total sum of y-values

c) Given $D=(V,A,c,s)$

and $A' \subseteq A$

- If A' contains a cycle 

  or $d^-_{A'}(v) > 1$ for some v

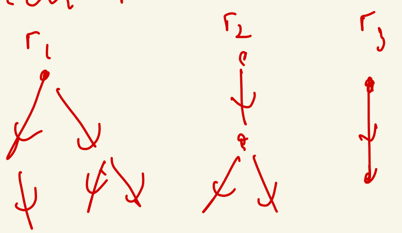
  or $d^-_{A'}(s) > 0$

  then there is no out-branching from s containing all arcs of A'

So assume $d^-_{A'}(v) \le 1$ $\forall v \ne s$ $d^-_{A'}(s) = 0$

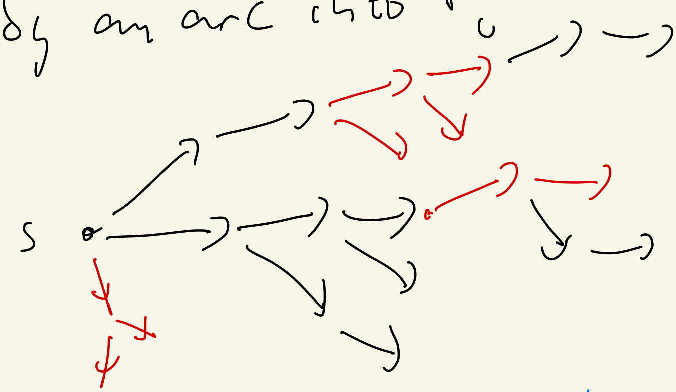
and that A' is acyclic

Then A' is a collection of out-trees



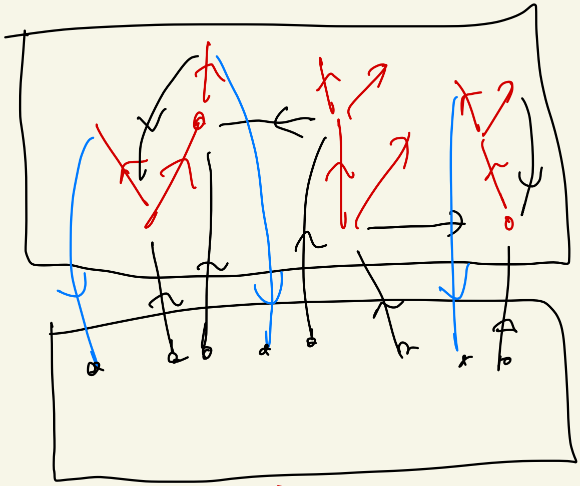
- let $r_1, r_2 \dots r_q$ be the roots of the nontrivial out-trees in $D' = (V, A')$

- if D has an out-branching B_s^+ s.t $A' \subseteq A(B_s^+)$, then B_s^+ must enter any out-tree $T_{r_i}^+$ when $r_i \neq s$ by an arc into r_i

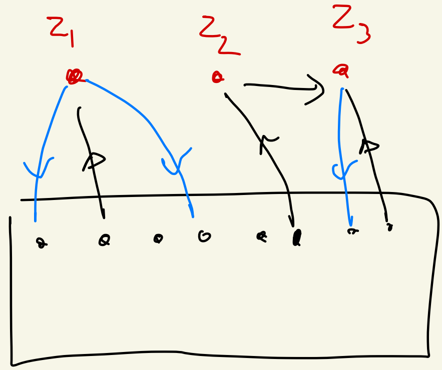


- contract each $V(T_{r_i}^+)$ to one vertex z_i whose out-neighbours is the set of out-neighbours of the nt $V(T_{r_i}^+)$ and whose in-neighbours are the in-neighbours of r_i in $V \setminus V(T_{r_i}^+)$

let D^* be the resulting digraph



$V \setminus \bigcup V(T_{r_i}^+)$



D^*

D has the desired out-branching B_u^+ (containing all arcs of A') if and only if D^* has an out-branching from s.

The cost of such an out-branching in D is the cost of the corresponding out-branching in D^* $+ \sum_{a \in A'} c(a)$ so minimizing cost of branching in D is the same as minimizing branching in D^*

d) $D=(V,A)$ strongly connected

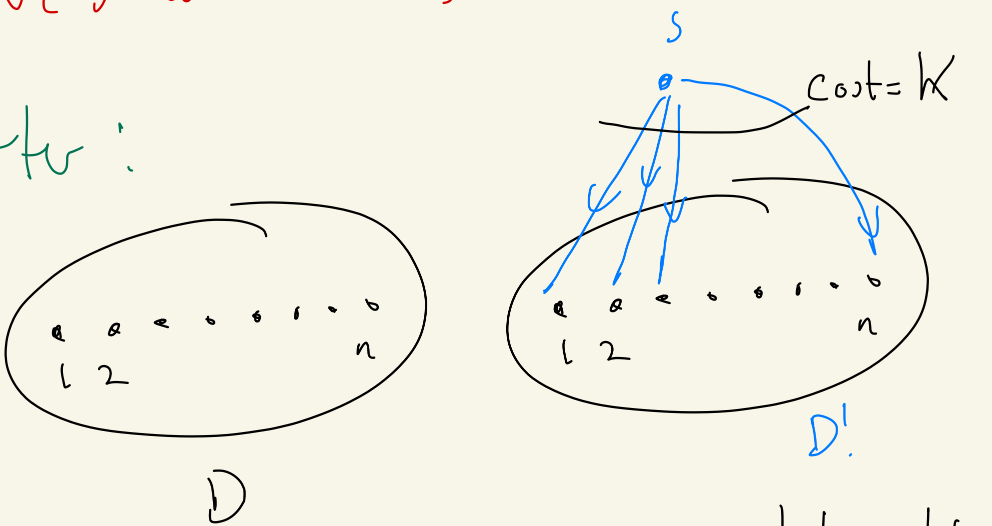
$c: A \to \mathbb{R}$ cost function

no root specified.

Goal: Find min cost out-branching when root can be arbitrary

Naive solution: try all possible roots

Better:



Choose K so large that no min cost branching from s will use 2 arcs out of s

B_s^+ optimal sol in D' $\Leftrightarrow$ $B_s^+ - \{s\}$ optimal in D

e) min cost out-branching with no root specified by contains all arcs of A' for some $A' \subseteq A$?

let D^* be as in c) and apply construction in d) to D^* instead of D