BJG 3.48 $\quad N = (V, A, \ell \equiv 0, u, b, c)$

i) assume: some arcs have $u_{ij} = \infty$

claim: There exists a finite ($x_{pq} < \infty$ $\forall pq \in A$) optimal feasible flow

$\Updownarrow$ There is no cycle W in N with $c(W) < 0$ and $u_{ij} = \infty$ $\forall ij \in A(W)$

$\Downarrow$ suppose there is a cycle W in N with $c(W) < 0$ and $u_{ij} = \infty$ $\forall ij \in A(W)$.
Then W is a cycle of $N(x)$ for every finite flow x so x cannot be optimal

$\Uparrow$ suppose that every cycle W of N with $c(W) < 0$ has at least one arc of finite capacity and let y be a finite flow which is feasible and has finite cost (we assume $|c_{ij}| < \infty$ $\forall ij \in A$)

let x be an arbitrary feasible flow in N
Then $\exists \tilde{y} \in N(y)$ s.t. $x = y \oplus \tilde{y}$ where $\tilde{y}$ is a circulation in $N(y)$

$$cx = cy + c\tilde{y} \geq cy + \sum_{i=1}^{r} \delta(W_i) c(W_i)$$ where

$\tilde{y}$ can be decomposed into cycle flows along $W_1, \ldots, W_r, \ldots W_p$ with $p \leq m$ and $c(W_i) < 0$ for $i \in [r]$ and $c(W_i) \geq 0$ for $i = r+1, \ldots, p$ and $\delta(W_i)$ is the ausumbus capacity of W_i in $N(y)$

let $R = \left|\sum_{i=1}^{r} \delta(W_i) c(W_i)\right|$. Then $R < \infty$ so

we can extend y to a finite optimal feasible flow x using at most R iterations of the cycle cancelling algorithm. The flow x will be finite on all arcs.

ii) let $A' = \{ij \in A \mid u_{ij} < \infty\}$ and $V' = \{v \mid b(v) > 0\}$

$$K = \sum_{ij \in A'} u_{ij} + \sum_{v \in V'} b(v)$$

Assume N has no cycle W with $c(W) < 0$ and $u_{ij} = \infty$ $\forall ij \in A(W)$.

Then by i) there exists a finite optimal flow x

Consider a flow decomposition of x into path flows along $P_1, \dots, P_r$ of values $\delta_1, \dots, \delta_r$ and cycle flows along $W_1, \dots, W_q$ of values $\beta_1 \dots \beta_q$

$$\sum_{i=1}^{r} \delta_i = \sum_{v \in V'} b(v)$$ as each P_i starts in V'

$$\sum_{j=1}^{q} \beta_j \leq \sum_{ij \in A'} u_{ij}$$ as each W_j intersects A'

Hence $x_{ij} \leq K$ for every arc $ij \in A$

BJG 3.79

Let $N = (V, A, \ell \equiv 0, u, b, c)$ have a feasible flow x':

Prove that N has a feasible flow x for which $|A_x| \le n-1$ when $n = |V|$ and

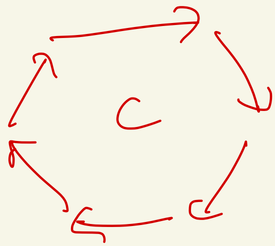
$A_x = \{ ij \mid 0 < x_{ij} < u_{ij} \}$

Remark: we may assume that all flows are netto flows ($x_{ij} \cdot x_{ji} = 0$ $\forall$ 2-cycle $i \to j \to i$)

We will show that we can make x s.t. A'_x has no cycle in the undirected sense

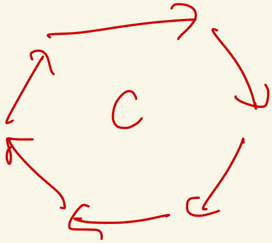
Suppose A_x has a cycle C:

Case I A_x has a directed cycle C



C is also a cycle in $N(x)$ and its residual capacity is

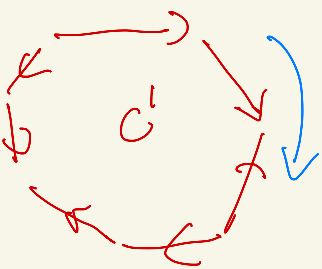
$\delta(C) = \min\{ u_{ij} - x_{ij} \mid ij \in A(C) \}$



let $x \Leftarrow x \oplus \delta(C) \cdot C$

then atleast one arc of C leaves A_X and no new arc enters A_X since we had $x_{ij} > 0$ on all arcs in A_X before. Thus $|A_X|$ decreased and x is still feasible

Case 2 A_X has an oriented cycle but no directed cycle.



fix a direction for traversal of C

let $\delta_f = \min\{u_{ij} - x_{ij} \mid ij \text{ forward cont orientation}\}$

$\delta_b = \min\{x_{ij} \mid ij \text{ backward cont orientation}\}$

$\delta \Leftarrow \min\{\delta_f, \delta_b\}$

Let C_f be the directed cycle in $N(x)$ which corresponds to traversing C in the fixed direction.

Then $\delta(C_f) = \delta$ so $x^* \leftarrow x \oplus \delta : C_f$ is feasible and $|A_{x^*}| < |A_x|$

Both in Case 1 and in Case 2 we could decrease the number of arcs ij with $0 < x_{ij} < u_{ij}$ so there exist a feasible flow x s.t. A_x has no cycle and thus $|A_x| \leq n-1$ $\square$.

Ahuja 8.7

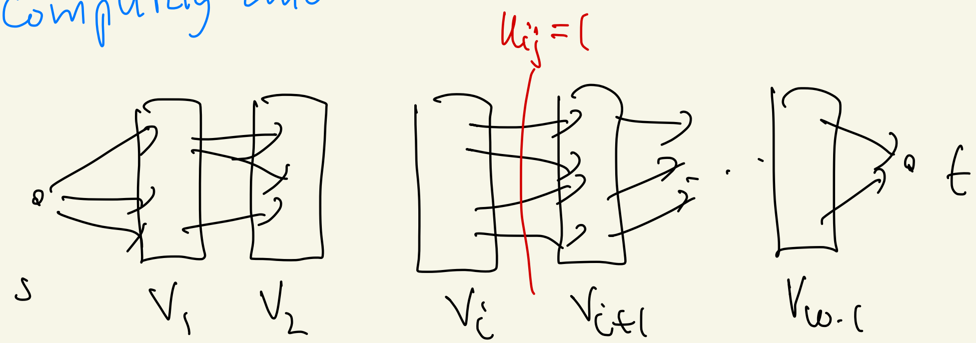
$N = (V \cup \{s,t\}, A, \ell \equiv 0, u)$ $\quad u_{ij} = 1$ when $i \neq s \wedge j \neq t$

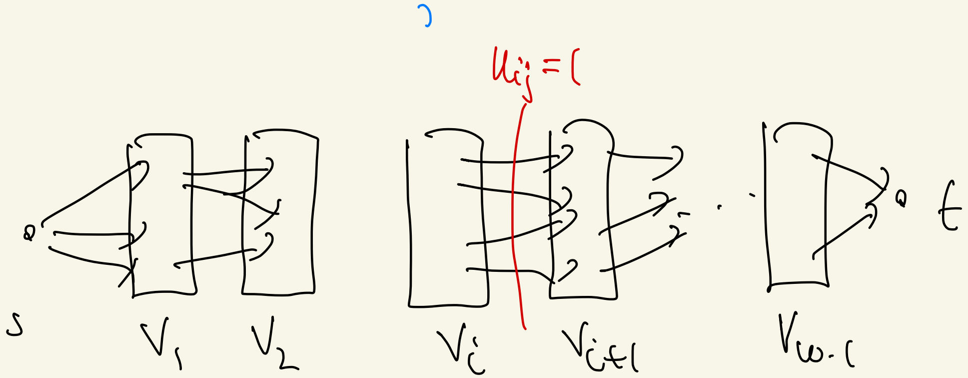
Q: does this change complexity of Dinic's algorithm?

A: we know from Exam 2020 prob. 4.3 that it does not change complexity. We prove it again below.

BJG lemma 3.7.2 still holds as we may assume all augmenting paths have capacity 1

[If $s \to t$ is an arc we start by filling it completely and consider N minus this arc]





Each of the cuts $(\{s\}\cup\ldots\cup V_i,\ V_{i+1}\cup\ldots\cup\{t\})$ have capacity at most $|V_i|\cdot|V_{i+1}|$ so

$$(*)\quad |x^*| \le |V_i|\cdot|V_{i+1}| \text{ for } i=1,2,\ldots,w-2$$

when x^* is a max flow

$$n=|V|\ge 2+\sum_{i=1}^{w-2}|V_i| \ge 2+\left\lfloor\frac{w-2}{2}\right\rfloor\cdot\sqrt{|x^*|}$$

$$\Downarrow$$

$$2n-4 \ge (w-2)\sqrt{|x^*|}$$

$$\Downarrow$$

$$w \le \frac{2n-4}{\sqrt{|x^*|}}+2 \qquad (\square)$$

So lemma 3.7.3 in BJG holds with small change proof of modified version of BJG Thm 3.7.4

- q = # phases in Dinic $J = \lceil n^{2/3} \rceil$
- $0 \equiv x^{(0)}, x^{(1)}, \ldots, x^{(q)}$ flows in algorithm

  and $k = |x^q|$ (value of max flow)
- choose j s.t $k - |x^{(j)}| > J$

  $k - |x^{(j+1)}| \leq J$
- Then value of max flow x^* in $N(x^{(j)})$ is at least J so

  $$\mathrm{dist}_{N(x^{(j)})}(s,t) \leq \frac{2n-4}{\sqrt{n^{2/3}}} + 2 = O(n^{2/3})$$
- Rest of proof is as original proof on page 123 in BJG.

If one more vertex may be incident to arcs of cap > 1 then we change (*) by a constant so alg still has same complexity.

Ahuja 8.8 $N = (V \cup \{s,t\}, A, \ell \equiv 0, u)$

$u_{ij} \in \{1,2,3,4\}$

Claim Dinic is still $O(n^{2/3} m)$:

Lemma 3.7.2: Each arc in at most 4 aug. paths before we delete it so $O(m)$ to find blocking flow

Lemma 3.7.3 Now $|x^*| \leq 4 |N_i| |N_{i+1}|$

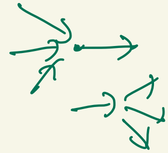
so $\min\{|N_i|, |N_{i+1}|\} \leq \frac{1}{2}\sqrt{|x^*|}$

$\Rightarrow$ same bound on dist(s,t) in O-notation

Remaining proof is same as BJG page 123

AH 8.10 Modified preflow-push als

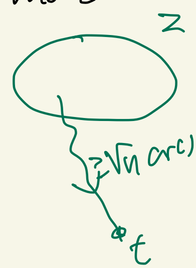
for simple unit capacity networks:



- No push/lift performed on v if $h(v) \geq \sqrt{n}$

a)
- Since $N(x)$ is also a unit cap network, there are no non saturating pushes.
- So at most $O(\sqrt{n})$ pushes along a given arc pq
- Also each vertex is lifted $O(\sqrt{n})$
- Hence total work until $h(v) \geq \sqrt{n}$ holds for every v which has $b_x(v) < 0$ is $O(\sqrt{n} \cdot m) + O(\sqrt{n} \cdot n) = O(\sqrt{n} m)$

b) <u>Claim:</u> if x is flow at termination of above process then at most $\sqrt{n}$ more units of flow can reach t



P: let $Z = \{v \mid b_x(v) < 0, v \neq t\}$

Each $v \in Z$ has distance at least $\sqrt{n}$ to t in $N(x)$ since h is a legal height function.

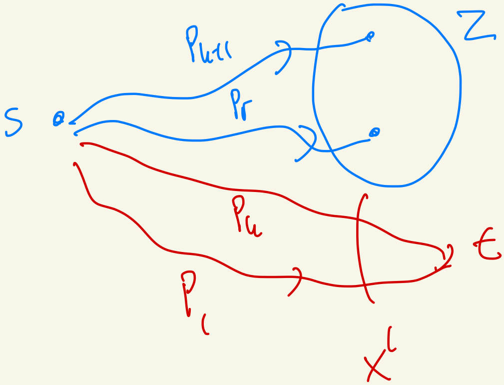
All flow from Z to t in $N(x)$ must follow vertex-disjoint paths (as $N(x)$ is unit capacity network) simple

let $P_1, P_2 \dots P_r$ be such paths

then $r \leq \frac{n}{\sqrt{n}} = \sqrt{n}$

So we can send at most $\sqrt{n}$ more units from Z to t.

c) let x be preflow when modified pfp stops.
As N is a unit cap Network we can decompose x into unit path flows along $P_1, P_2 \dots P_k, P_{k+1} \dots P_r$ and some cycles, where $P_1 \dots P_k$ are (s,t)-paths
we can find this decomposition in linear time $O(n+m)$ since when we extract a path or cycle, all its arcs become empty



s
P_{u+1}
P_r
Z
P_u
P_l
t
X^l

- Let x' be the (s,t)-flow formed by the union of the path flows along $P_1, P_2, \ldots P_k$

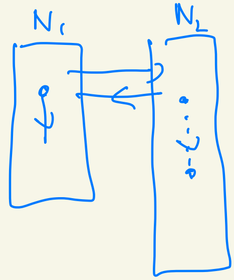
- Then the maximum flow value in $N(x')$ is at most $\sqrt{n}$ by b)

- So we can find a max flow y in $N(x')$ in time $O(\sqrt{n}\, m)$

- adding y to x' gives a maximum flow in N.

Ahuja 8.18

semi bipartite



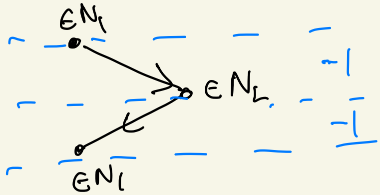
assume $n_1 \leq n_2$ show that

we can modify generic pfp so

that it runs in time $O(n_1^2 m)$ time on semi bipartite networks.

Consider the algorithm from Section 8.3 when all

pushes came in pairs



now we also allow

analysis still works as an unsaturating push

now decreases $\Phi = \sum_{i \text{ active}} h(i)$ by at least 1

instead of at least 2 as in 8.3

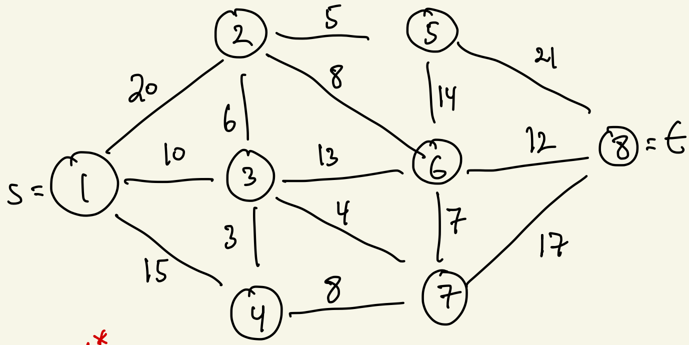
Hence the analysis from 8.3 still works

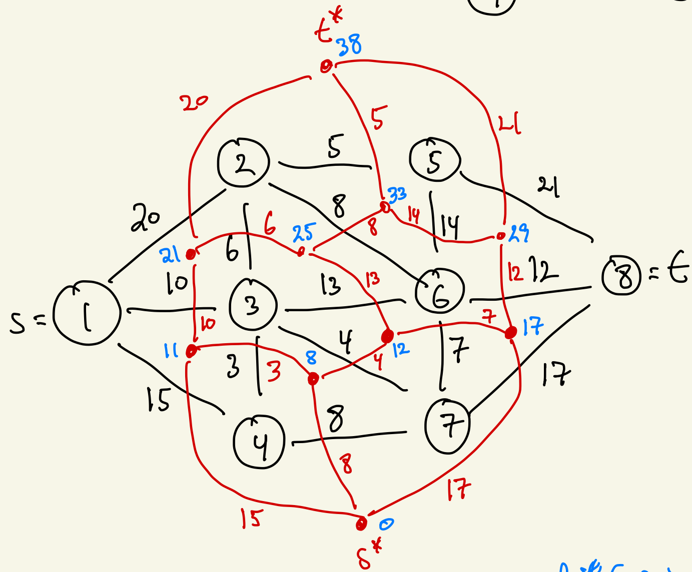
and we see that the algorithm

runs in time $O(n_1^2 m)$ as before.

Ahuja 8.21

Goal: find a maximum flow in the undirected network given here, using the algorithm of section 8.4.



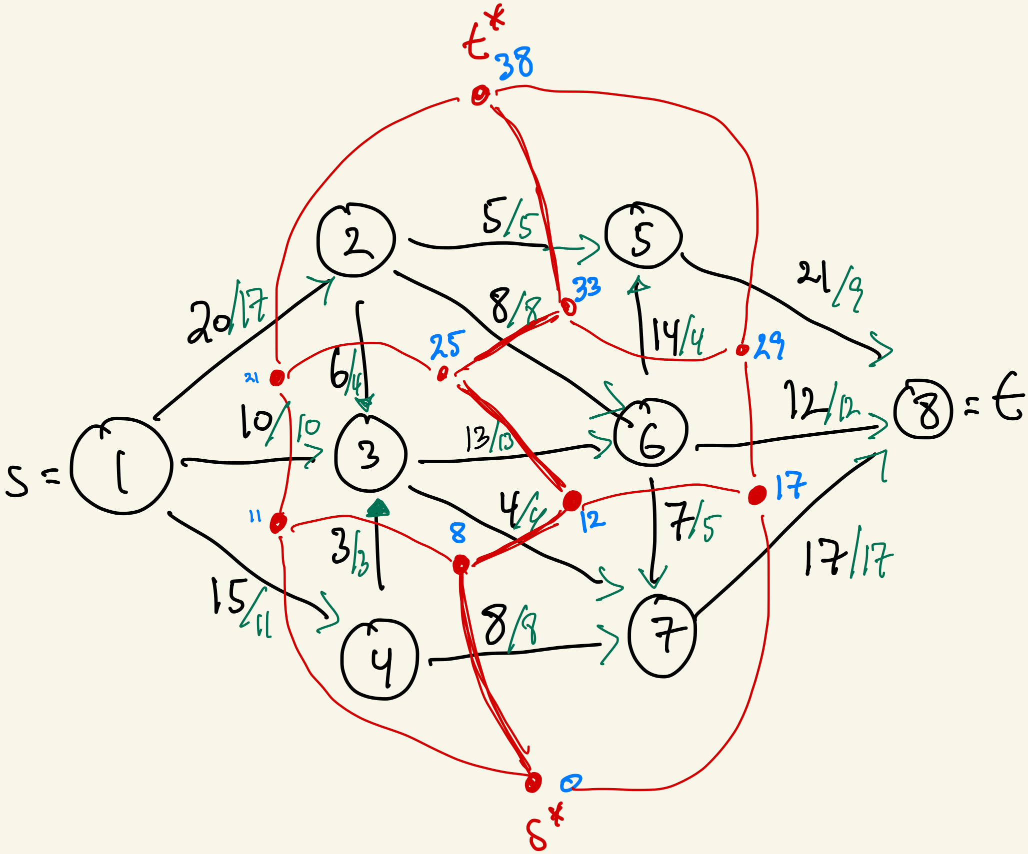


Dual of G

$d^*(v) = d(s^*, v)$ indicated in blue $\quad d^*(t) = 38$

so we know that the max flow value is 38

and we get such a flow by setting $x_{ij} = d^*(j) - d^*(i)$ when this is positive!



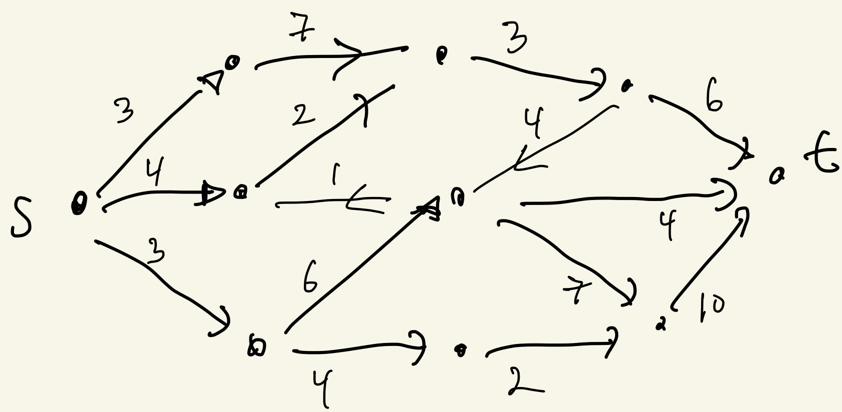
s = 1
8 = t
t*
s*
20/17
5/5
21/9
8/8
6/4
14/4
10/10
13/13
12/12
4/4
3/3
7/5
15/11
8/8
17/17
38
33
25
29
21
11
8
12
17
0

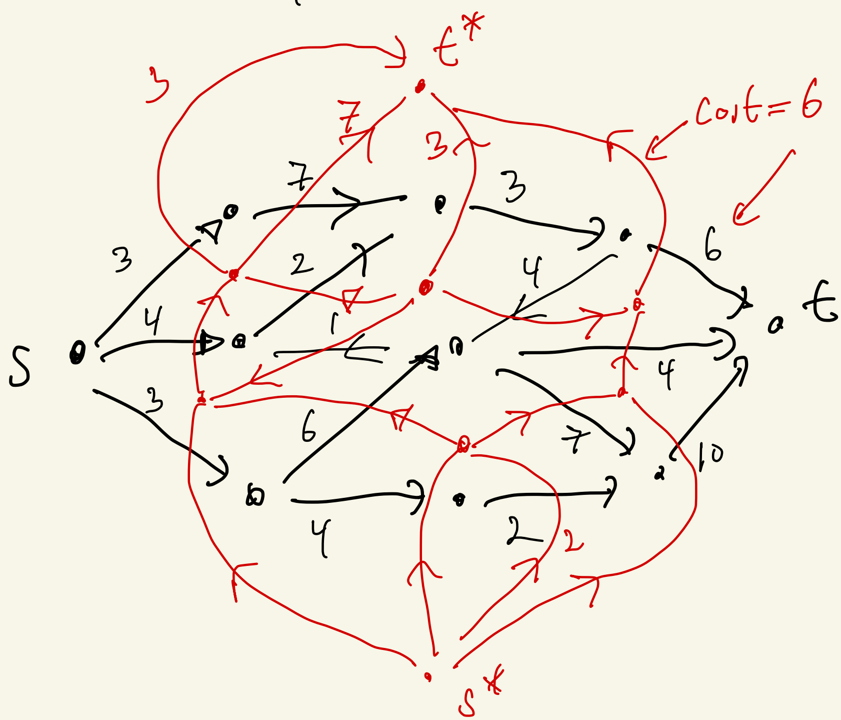
Ahuja 8.22

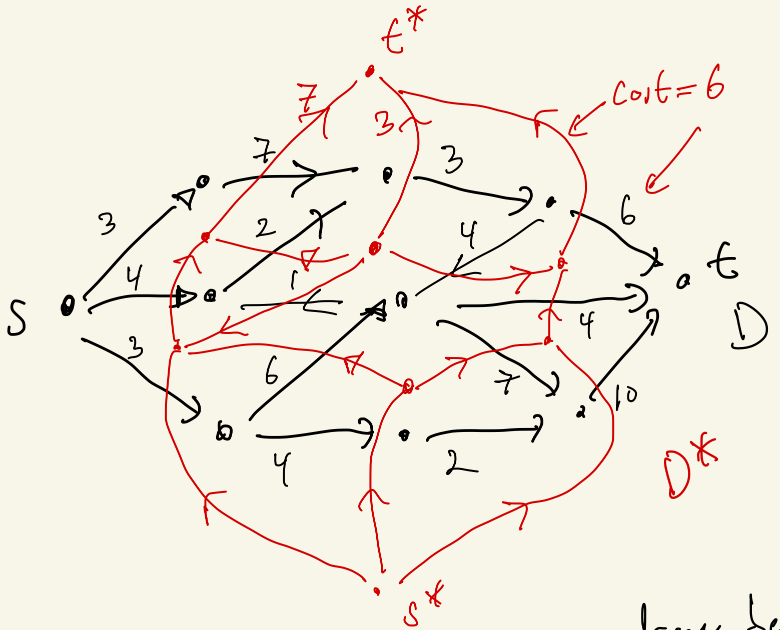
G is a planar (s, t) directed network

(s and t are on the outer face)

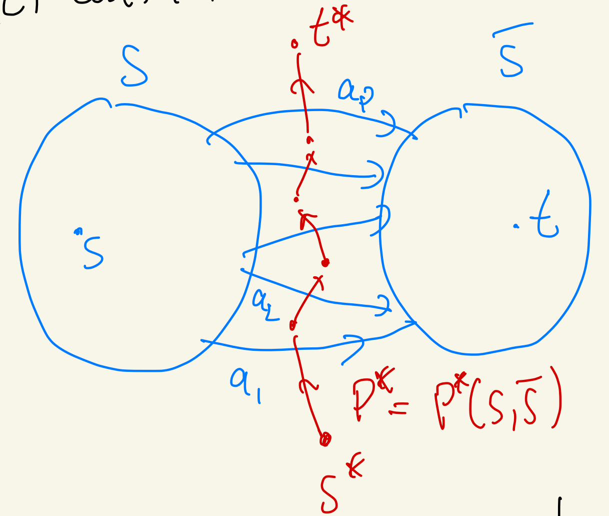
u_{ij} shown







<u>Claim</u> there is a 1-1 correspondance between (s,t)-cuts in D and (s^*,t^*)-paths in D^*



$a_1, a_2 \cdots a_p$ are the $S \to \bar{S}$ arcs ordered after position in embedding

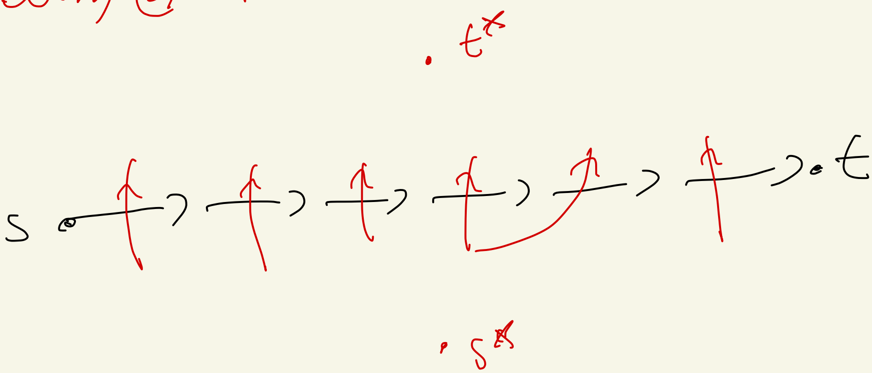
Every (s^*,t^*)-path P^* gives a cut, namely the arcs in D intersected by the arcs in P^* and

$\text{cost}(P) = u(S,\bar{S})$

Ahuja 8.23

<u>claim</u>: min # of arcs in a directed (s,t)-path in D
= max # arc-disjoint (s,t)-cuts in D

P: consider the dual D^* of D and note that every (s,t)-path in D defines an (s^*,t^*)-cut in D^*



Hence

length of a shortest (s,t)-path in D

= min # arcs in an (s^*,t^*)-cut in D^*

= max # arc-disjoint (s^*,t^*)-paths in D^* ← by Menger's theorem

= max # arc-disjoint (s,t)-cuts in D

Ahuja 8.13

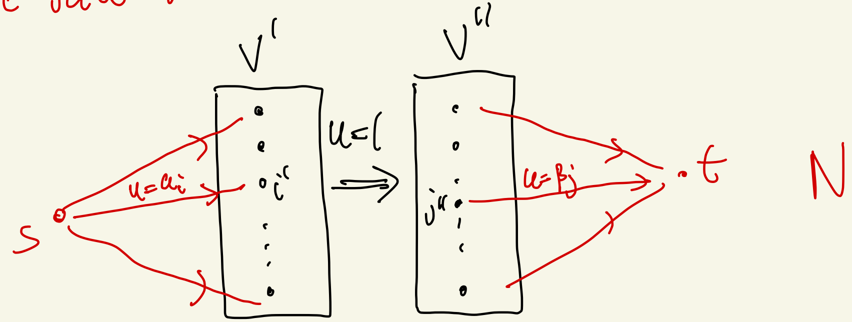
Given integers $\alpha_1, \dots \alpha_n, \beta_1, \dots, \beta_n$ such that

$$\sum_{i=1}^{n} \alpha_i = \sum_{j=1}^{n} \beta_j \quad (*)$$

Question: Does there exist a digraph D on n vertices $v_1, v_2, \dots, v_n$ with $d^+_D(v_i) = \alpha_i$ and $d^-_D(v_i) = \beta_i \ \forall i \in [n]$

Solution: Note that D exists if and only if the complete digraph $\overleftrightarrow{K}_n$ ($ij \in A \ \forall i \neq j$) has D as a subdigraph.

We saw how to check this in BJG Thm 3.11.5



N has an (s,t)-flow of value $\sum_{i=1}^{n} \alpha_i = \sum_{j=1}^{n} \beta_j$

$\Updownarrow$

D exists

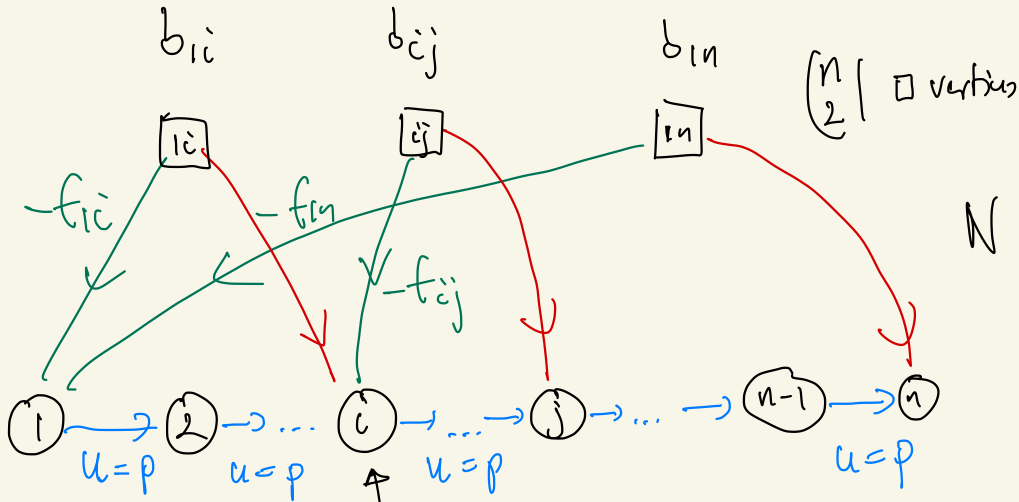
Ahuja 9.13 $\approx$ BJG pages 130-131

b_{ij} = # passengers who wish to go from i to j

f_{ij} = ticket price for travel i to j

p = capacity of plane

Goal: maximize income from accepted customers

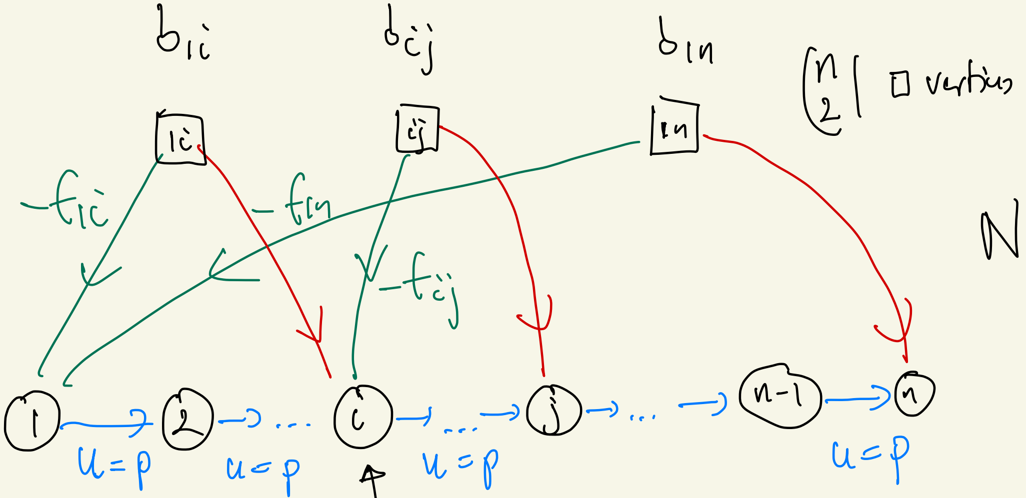


$$b_i = -\sum_{j=1}^{i-1} b_{ji}$$

green arcs $\Longleftrightarrow$ passengers taken

red arcs $\Longleftrightarrow$ passengers not taken

blue arcs model the capacity of the plane



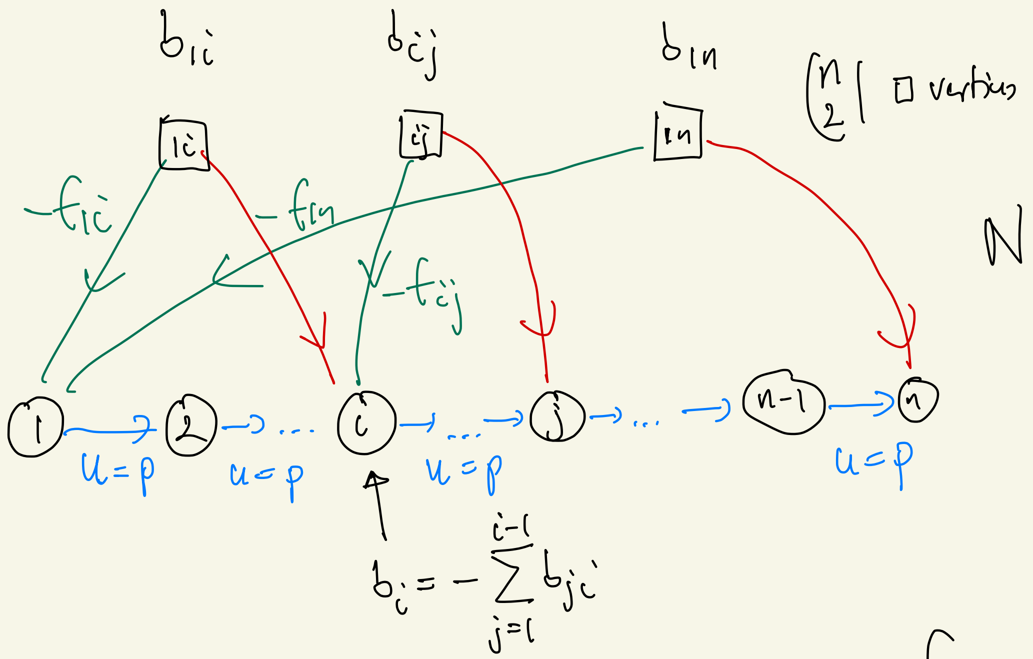
$$b_i = -\sum_{j=1}^{i-1} b_{ji}$$

claim: $-1 \cdot \min\{cx \mid x \text{ feasible flow in } N\}$

= max income we can obtain

$\leq$: each feasible flow x corresponds to legal passenger assignment and $x_{(ij)i}$ is the number of passengers from i to j that is taken.

So $cx = \sum_{1\leq i<j\leq n} x_{ij} f_{ij}$ = earnings from setting $t_{ij} = x_{(ij)i}$



$\geq$ Let t_{ij} denote # of passengers from i to j that we accept. Then we obtain a feasible flow x by setting

$$x_{(ij)i} = t_{ij}, \quad x_{(ij)j} = b_{ij} - t_{ij}, \quad x_{q\,q+1} = \sum_{\substack{a \leq q \\ b \geq q+1}} t_{ab}$$

and

$$-cx = \sum_{1 \leq i < j \leq n} t_{ij} f_{ij} = \text{income from assignment } t_{ij} \quad 1 \leq i < j \leq n$$

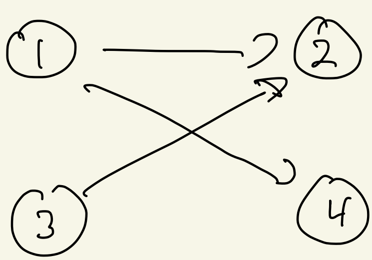
Ahuja 9.14 Tanker scheduling with startup costs.

Same data as in application 6.6

shipment	1	2	3	4
profit	10	10	3	4

<u>Problem</u> Assume it cost 5 units to use a ship and we do not have to handle all shipments.

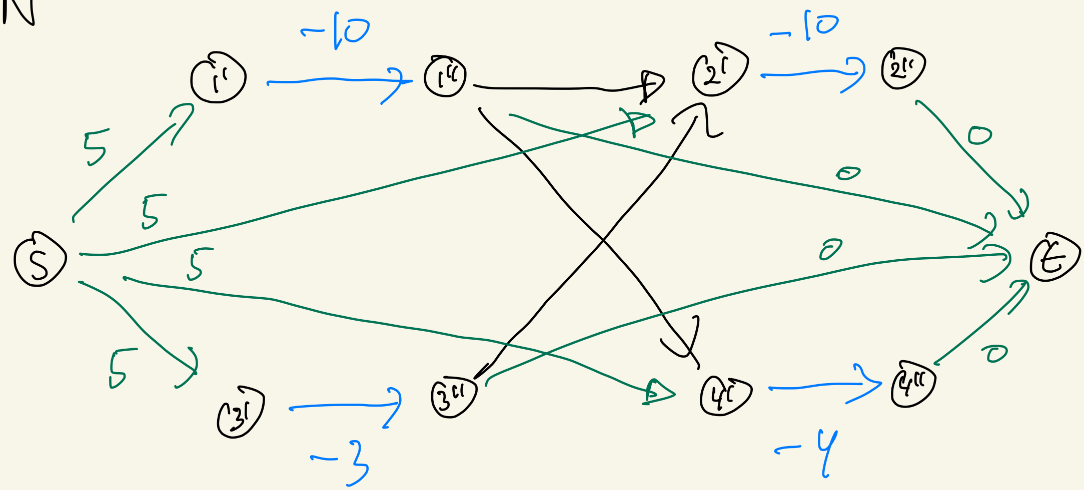
<u>Goal</u>: maximize profit minus costs for ships.



$i \to j$

one ship can handle i and then j

N



all arcs have capacity 1

<u>Claim</u> (s,t)-flow of min cost gives a solution

P: We know (from earlier) that a solution with k ships corresponds to a flow of value k in N and the cost of such a flow is exactly the profit for that solution

Problem: How do we find an (s,t)-flow of minimum cost efficiently (value not prespecified)

Solution: use Buildup method

$x \leftarrow 0$ (zero flow)

while $\exists (s,t)$-path P with $c(P) < 0$ in $N(x)$ do
augment x along P

Let x be flow after termination of the loop

Then

- $\forall (s,t)$-path P in $N(x)$: $c(P) \geq 0$
- $\forall$ cycle W in $N(x)$ $c(W) \geq 0$ $\boxed{W}$

Claim $cx \leq cx' \quad \forall (s,t)$-flow x'

P: if $|x'| \leq |x|$ then $cx \leq cx'$ as x was obtained from a min cost flow x'' with $|x''| = |x'|$ by augmenting along zero or more paths of negative cost assume $|x| < |x'|$. Then there exist $\tilde{x} \in N(x)$ such that $x' = x \oplus \tilde{x}$ and

$$cx' = cx + c\tilde{x}$$
$$\geq cx$$

since $\tilde{x}$ is a sum of path and cycle flows in $N(x)$, all of which have non-negative cost by [□].

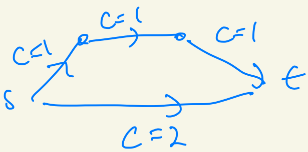
Ahuja 9.39

$N=(V,A,\ell\equiv 0,u,b,c)$ c_{kl} minimum among all costs
c_{pq} maximum - - -

Q1 is it possible that no min cost flow has $x_{kl}>0$?

Q2 is it possible that every min cost flow has $x_{pq}>0$

Answer: Yes to both!



Every min cost (s,t)-flow of value 1 sends one unit along $s\to t$

Ahuja 9.40

a) $N=(V,A,\ell\equiv 0,u,b,c)$ all $b(v), u_{ij}$ even

Q: does there exist optimal feasible flow x s.t. x_{ij} is even $\forall ij \in A$

Yes: $N \to N' = (V,A,\ell\equiv 0,\frac{u}{2},\frac{b}{2},c)$

x' optimal feasible flow in N'

$x \in 2x'$ optimal and feasible in N

if π is a potential s.t. $c^{\pi}_{ij}\geq 0$ $\forall ij \in N'(x')$ so is x
via same π as $c^{\pi}_{ij}\geq 0$ $\forall ij \in N(x)$ optimal

b) $N = (V, A, \ell \equiv 0, u, b \equiv 0, c)$

u_{ij} even $\quad y \in N$ feasible circulation

Same method as in a)

shows that there exists a

feasible circulation x with x_{ij} even

for all arcs ij

Ahuja 9.41 $\quad N = (V, A, l \equiv 0, u, b, c)$

x^* optimal solution and
π potential s.t $c_{ij}^{\pi} \geq 0 \quad \forall ij \in N(x^*)$

Let $N^\circ \subseteq N(x^*)$ consist of arc
with reduced cost $= 0$

- Suppose x' is another optimal feasible flow in N.
- Then $x' = x^* \oplus y$ where y is a circulation in $N(x^*)$

and $c^{\pi} x' = c^{\pi} x^* + c^{\pi} y$ so

$c^{\pi} y = 0$ and y decomposes into cycle flows in $N(x^*)$

Since $c_{ij}^{\pi} \geq 0$ on all arcs each cycle W
in such a decomposition has $c^{\pi}(W) = 0$