Network = directed graph

$D = (V, A)$

+ some extra parameters

$N = (V, A, l, u, b, c)$

$$b(i)\quad \overset{i}{\circ} \xrightarrow[u_{ij},\ c_{ij}]{l_{ij}\quad x_{ij}} \overset{j}{\circ}\quad b(j)$$

A flow in N: $x \to A \to \mathbb{R}_+ \cup \{0\}$

$0 \leq l_{ij} \leq u_{ij} \quad \forall ij \in A$

cost function c_{ij} defined on all pairs $i, j \in V$

$c_{ij} = -c_{ji} \quad \forall (i,j) \in V \times V$

balance vector $b : V \to \mathbb{R}$

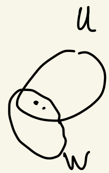
$$\sum_{v \in V} b(v) = 0 \tag{3.3}$$

functions on $V \times V$

if $f : V \times V \to \mathbb{R}$

$$f(U,W) = \sum_{\substack{i \in U \\ j \in W}} f_{ij} \tag{3.4}$$

$U, W \subseteq V$



A flow x in N:

$$x : A \to \mathbb{R}_+ \cup \{0\} = \mathbb{R}_0$$

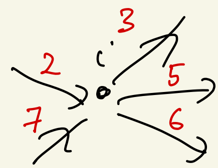
if $ij \notin A$ then define $x_{ij} = 0$

A flow is an integer flow if

$$x_{ij} \in \mathbb{Z}_0 \quad \forall ij \in A$$

Balance vector of a flow x:

$$b_x(i) = \sum_{ij \in A} x_{ij} - \sum_{ji \in A} x_{ji}$$



$b(i) = 5$

Classification of vertices wrt a flow x:

i is a **source** if $b_x(i) > 0$

i is a **sink** if $b_x(i) < 0$

i is **balanced** if $b_x(i) = 0$

x is feasible in $N = (V, A, \ell, u, b, c)$

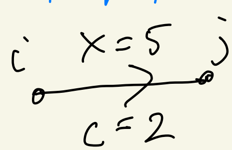
if

$$\ell_{ij} \leq x_{ij} \leq u_{ij} \quad \forall ij \in A$$

$$b_x(i) = b(i)$$

Cost of x in $N = (V, A, \ell, u, b, c)$

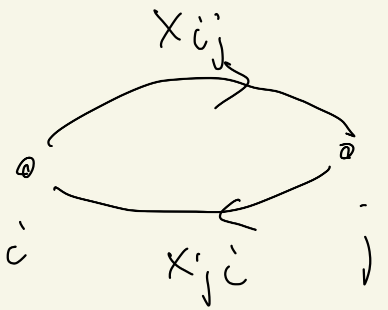
$$c^T x = \sum_{ij \in A} c_{ij} x_{ij}$$

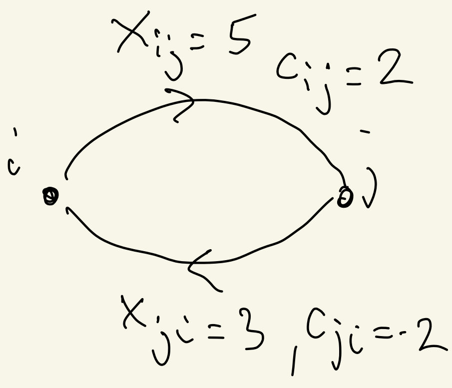


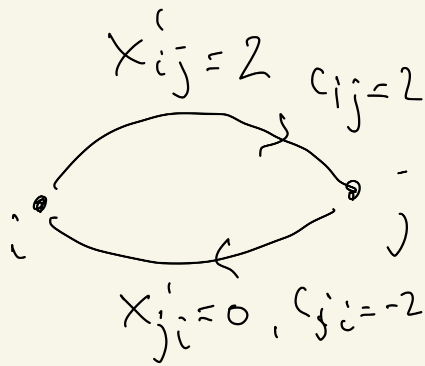
Assumption

Every flow that we consider is a Netto flow

$$x_{ij} \cdot x_{ji} = 0 \quad \forall ij \in A$$

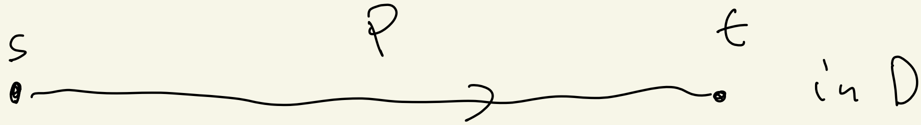


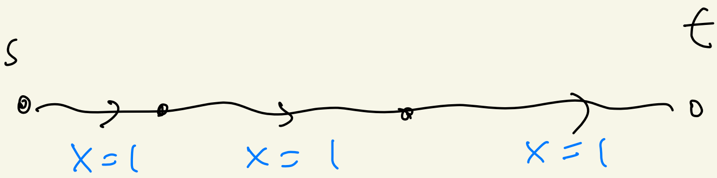


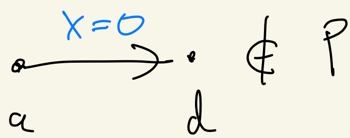


cost between i and j
4

cost between i and j
4







$$b_x(v) = \begin{cases} 1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$$

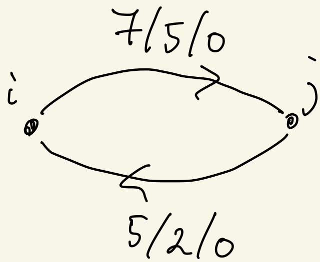
[(s,t)-flow of value 1]

Residual network with respect to a flow x in $N=(V,A,l,u,c)$

$$N(x) = (V, A(x), \bar{l} \equiv 0, r, c)$$

$$r_{ij} = (u_{ij} - x_{ij}) + (x_{ji} - l_{ji})$$

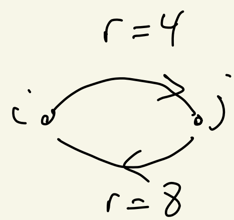
$$ij \in A(x) \iff r_{ij} > 0$$



$u/x/l$

$$r_{ij} = (7-5) + (2-0) = 4$$

$$r_{ji} = (5-2) + (5-0) = 8$$



Various flow types/models

- (s,t)-flows:

$$b_x(s) = -b_x(t), \quad b(v) = 0 \;\; \forall v \neq s,t$$

- Circulation

$$b_x(v) = 0 \quad \forall v \in V$$

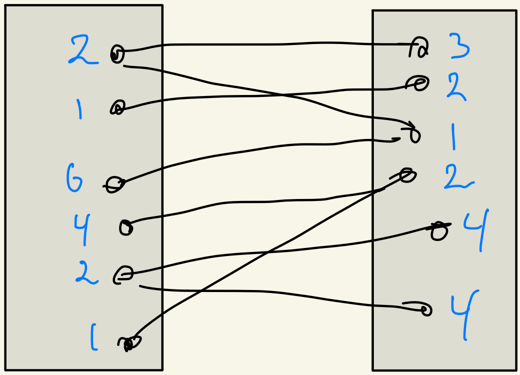
- general balances

only require $\sum_{v \in V} b_x(v) = 0$

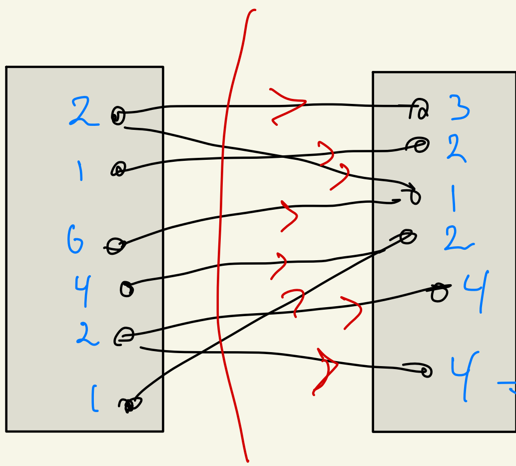
- bounds and costs on vertices

$$\ell^x(v), u^x(v), c^x(v)$$

v



$u \equiv 1,\ \ell \equiv 0$



want an integer flow x with $b_x(v) = b(v)$

Eliminating lower bounds

example of when lower bounds are relevant:

if we insist on certain arcs being used

$$N \qquad\qquad\qquad\qquad N'$$

$$\begin{tikzcd} \overset{b(i)}{i} \arrow[r, "u_{ij}", "l_{ij}>0"'] & \overset{b(j)}{j} \arrow[r] & \overset{b(i)-l_{ij}}{\bullet} \arrow[r, "l'_{ij}=0"'] & \overset{b(j)+l_{ij}}{\bullet} \end{tikzcd}$$

$$u'_{ij} = u_{ij} - l_{ij}$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad\qquad 0 \le x'_{ij} \le u'_{ij}$$

$$x_{ij} = x'_{ij} + l_{ij}$$

$$c^T x = c^T x' + l_{ij} \cdot c_{ij}$$

Conclusion we can eliminate

all lower bounds in time

$$O(n+m) \qquad n = |V|, \quad m = |A|$$

and we can convert any flow in N' to a flow in N in time $O(n+m)$

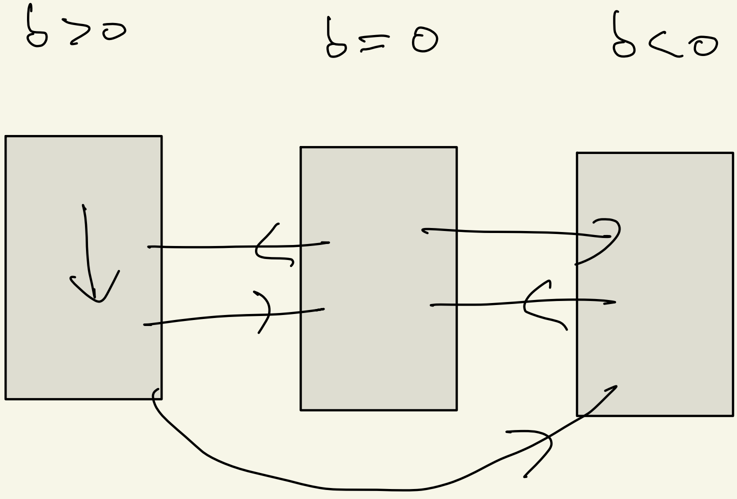
General balances → (s,t)-flows

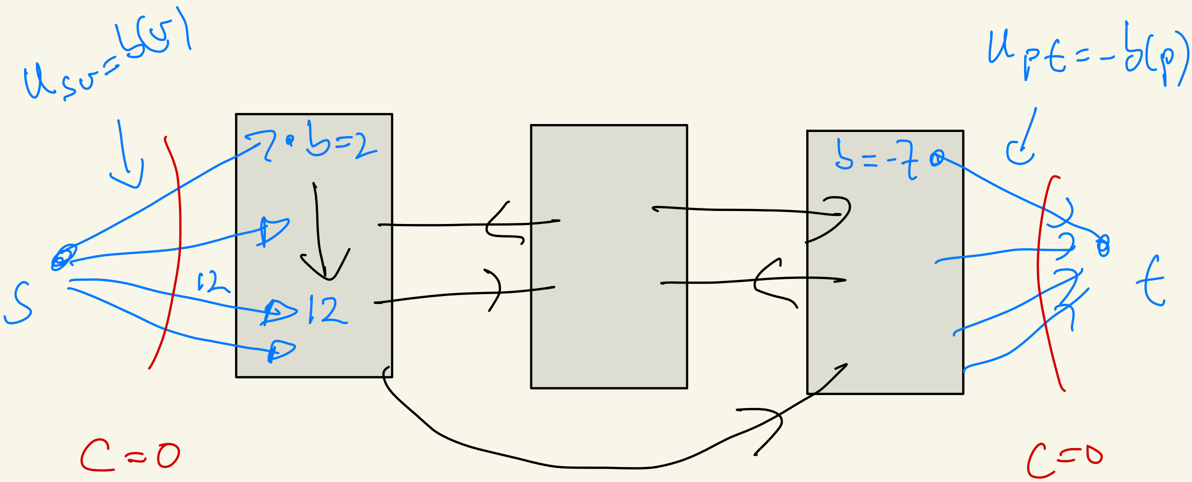
$$N = (V, A, \ell \equiv 0, u, b, c)$$

$$\downarrow$$

$$\hat{N} = (V \cup \{s,t\}, A \cup F, \hat{\ell} \equiv 0, \hat{u}, \hat{b}, \hat{c})$$

require $\hat{b}(v) = 0 \quad \forall v \neq s, t$





Every feasible flow x in N corresponds to a flow $\hat{x}$ in $\hat{N}$ and conversely <u>and</u> $c^T x = \hat{c}^T \hat{x}$

$$\hat{b}(s) = \sum_{\substack{v \in V \\ b(v) > 0}} b(v) \qquad \hat{b}(t) = \sum_{\substack{v \in V \\ b(v) < 0}} b(v)$$

$$\hat{b}(v) = 0 \quad \forall v \neq s, t$$