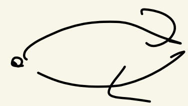
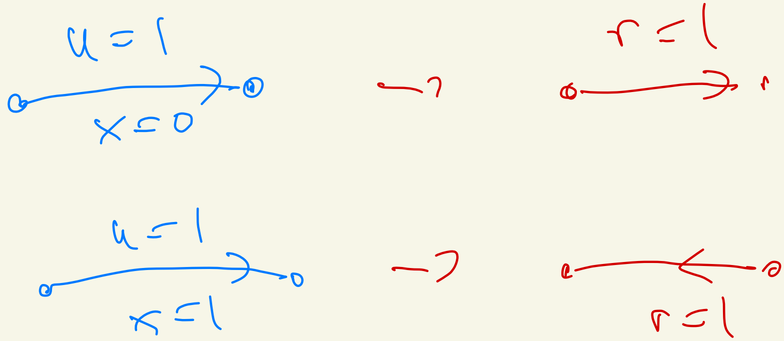
3.7 in BJG

assumption no  in N

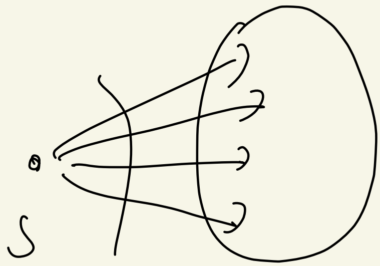
$N = (V \cup \{s,t\}, A, \ell \equiv 0, u \equiv 1)$

always assume that x is an integer flow

Lemma 3.7.1 if x is (s,t)-integer flow in N unit cap network then $N(x)$ is also a unit cap Netw.

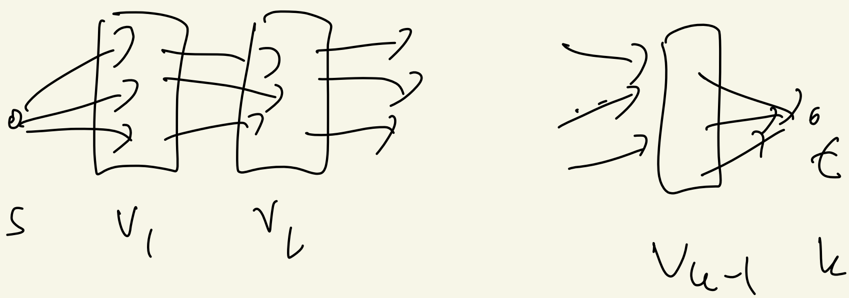


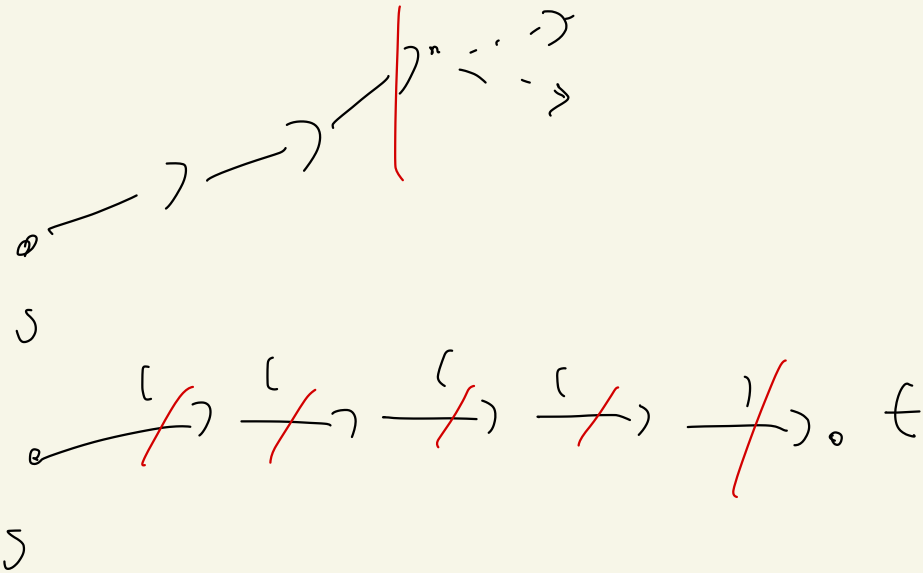
Observation Ford-Fulkerson alg finds a maxflow in a unit cap N. in time $O(nm)$!



$$\sum u_{sj} \leq n-1$$

Lemma 3.7.2 Dinic's alg finds a new blocking flow in time $O(m)$



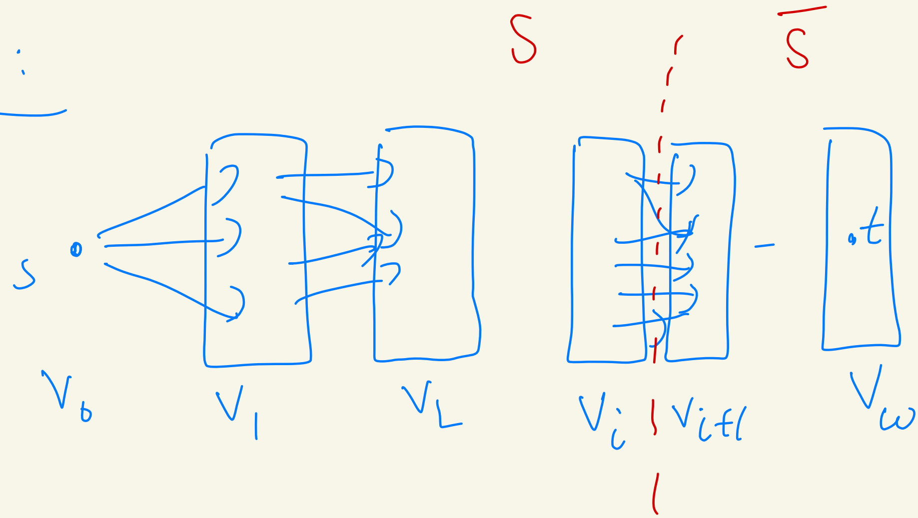


total work to find a blocking flow is $O(m)$.

Lemma $N = (V \cup \{s,t\}, A, l \equiv 0, u \equiv 1)$

suppose x^* is a max flow in N
then $\text{dist}_N(s,t) = 2n/\sqrt{|x^*|}$

P:



the arcs from V_i to V_{i+1} are the arcs crossing the cut $(S, \bar{S})$

the capacity is at most $|V_i||V_{i+1}|$

Hence $|x^*| \le |V_i||V_{i+1}|$ for all

$i = 0, 1, 2 \ldots \omega - 1$

Hence $|X^*| \le |V_i||V_{i+1}|$ for all

$i = 0, 1, 2 \ldots \omega - 1$

$$n = |V| \ge \sum_{i=0}^{\omega} |V_i|$$

$$\ge \sqrt{|X^*|} \left\lfloor \frac{\omega+1}{2} \right\rfloor$$

$$\Downarrow$$

$$\boxed{\omega \le \frac{2n}{\sqrt{|X^*|}}}$$

Thm 3.7.4 Dinic's alg finds a max flow in $N = (V \cup \{s,t\}, A, \ell \equiv 0, u \equiv 1)$ in time $O(n^{2/3} m)$

Let q be the number of phases (finding new blocking flow in $N(x)$ and updating x)

Denote by $0 \equiv x^{(0)}, x^{(1)}, x^{(2)}, \ldots x^{(q)}$ the corresponding flows in N

$x^{(q)}$ is a max flow in N.

Let $J = \lceil n^{2/3} \rceil$ and $K = |x^{(q)}|$

sufficient to show that $q \in O(n^{2/3})$

let $J = \lceil n^{2/3} \rceil$ and $K = |X^{(q)}|$

sufficient to show that $q \in O(n^{2/3})$

<u>Case 1</u> $K \leq J$

then we are done since each blocking flow has value ≥ 1 so $\leq J$ phases so $O(n^{2/3} m)$

<u>Case 2</u> $K > J$

$X^{(0)}, X^{(1)}, \ldots, X^{(j)}\ X^{(j+1)} \ldots X^{(q)}$

j chosen so that $|X^{(j)}| < K - J$

and $|X^{(j+1)}| \geq K - J$

$x^{(0)}, x^{(1)}, \ldots, x^{(j)}, x^{(j+1)}, \ldots, x^{(q)}$

j chosen so that $|x^{(j)}| < K - J$

and $|x^{(j+1)}| \geq K - J$

The value of a max flow in

$N(x^{(j)})$ is $K - |x^{(j)}| > J = \lceil n^{2/3} \rceil$

Lemma 3.7.3 $\Rightarrow$

$$\underline{\text{dist}_{N(x^{(j)})}}(s,t) \leq 2n / \sqrt{n^{2/3}} = \underline{2n^{2/3}}$$

so $j \leq 2n^{2/3} \Rightarrow$ we spend $O(n^{2/3} m)$
time to get $x^{(j)}$

and $q - (j+1) \leq J \Rightarrow$ also here $O(n^{2/3} m)$
time $\Rightarrow$ $O(n^{2/3} m)$ in total