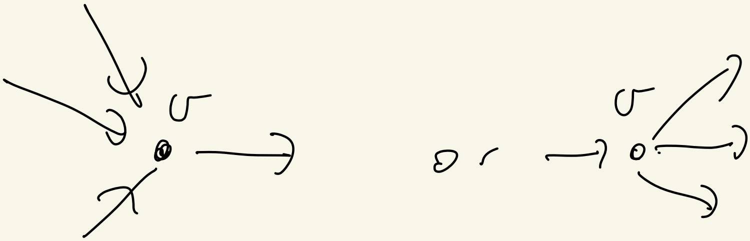
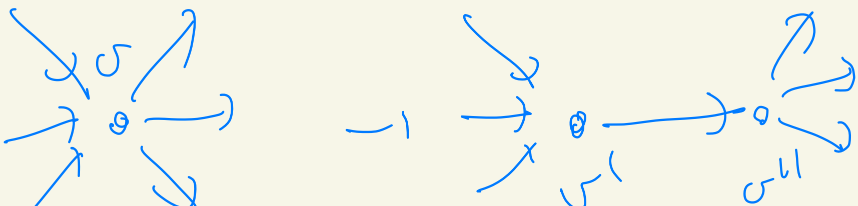
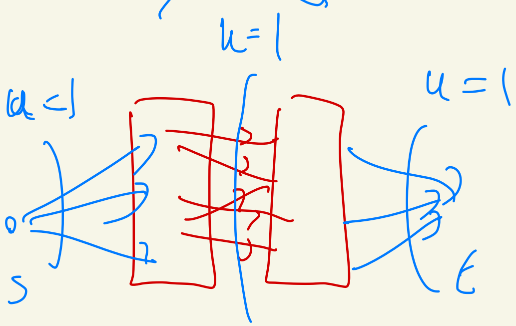
Definition a network is simple if

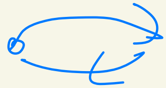


$$\min\{d^+(v), d^-(v)\} \leq 1 \quad \forall v \in V$$

vertex splitting produces simple networks





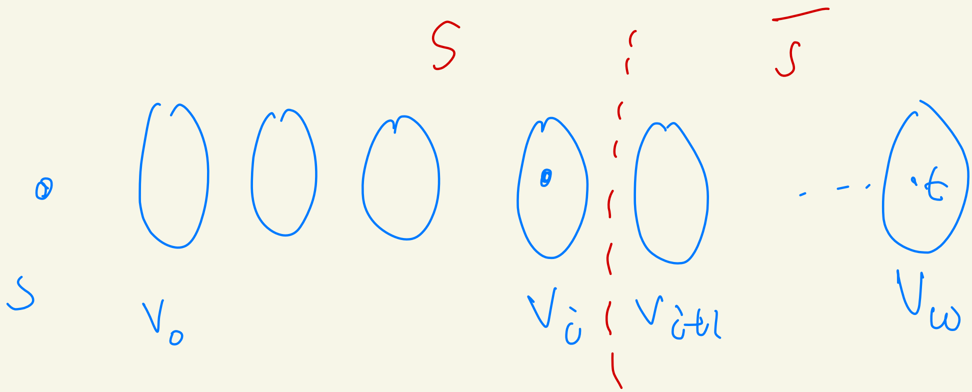
assume  in N

we want to show:

<u>**Thm**</u> if $N = (V \cup \{s,t\}, A, \ell \equiv 0, u \equiv 1)$ is simple, then Dinic's alg finds a max flow in time $O(\sqrt{n}\, m)$

<u>**Lemma**</u> 3.7.5 let x^* be a max flow in N then

$$\mathrm{dist}_N(s,t) \le \frac{n}{|x^*|}$$



$$|X^*| \le \min\{|V_i|, |V_{i+1}|\} \quad i=0,\dots,\omega-1$$

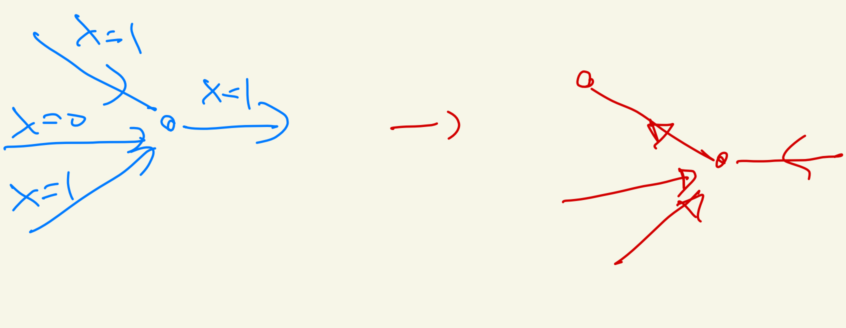
$$n = |V| > \sum_{i=1}^{\omega-1} |V_i| \ge |X^*|(\omega-1)$$

$$\Downarrow$$

$$\omega - 1 < \frac{n}{|X^*|} \quad \text{so} \quad \omega \le \frac{n}{|X^*|}$$

<u>Lemma</u> 3.7.6 if N is simple unit cap and x feasible in N then $N(x)$

(i) a simple unit cap network



proof of theorem

$\tilde{x}^{(1)}$ blocking in $N(x^{(1)})$ $\quad x^{(2)} = x^{(1)} \oplus \tilde{x}^{(1)}$

$$0 \equiv x^{(0)},\ x^{(1)},\ x^{(2)},\ \ldots\ldots,\ x^{(q)}$$

q phases each finding a new blocking flow

Let $k = |x^{(q)}|$ (value of a max flow)

$$Z = \lceil \sqrt{n} \rceil$$

Look at phases

$$x^{(0)}, x^{(1)}, \ldots, x^{(j)}, \underbrace{x^{(j+1)}, \ldots, x^{(q)}}_{\leq Z \text{ phases by } (*)}$$

j is chosen s.t. $\quad k - |x^{(j)}| > Z \quad (*)$

and $\quad k - |x^{(j+1)}| \leq Z$

bounding j:

In $N(x^{(j)})$ the max flow value is larger than Z since

$$k - |x^{(j)}| > Z$$

so $\mathrm{dist}_{N(x^{(j)})}(s,t) \le \dfrac{n}{Z} \le \sqrt{n}$

$$x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(j)}$$

distance increases by at least 1

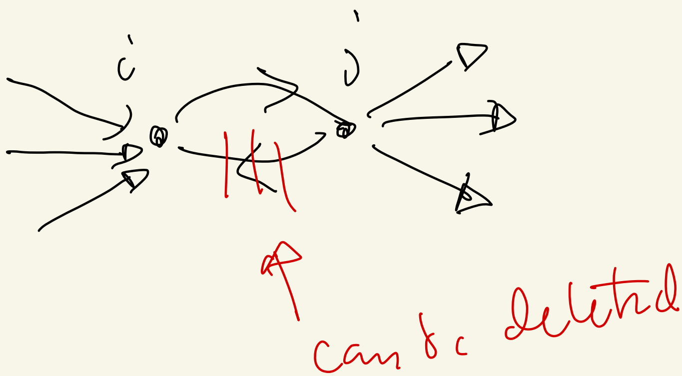
$j \le \sqrt{n}$

Conclusion $q \le \sqrt{n} + \sqrt{n} \in O(\sqrt{n})$

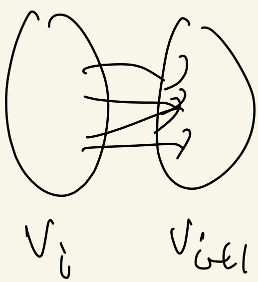
so algorithm is $O(\sqrt{n}m)$ □

Can when we do have $i \rightleftarrows j$

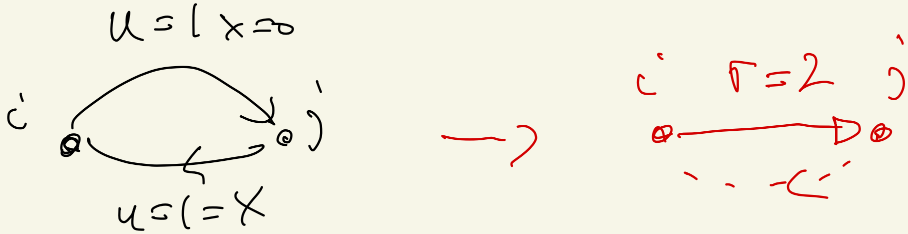
if N is simple and unit cap.
then we can delete $i \to j$ or $j \to i$

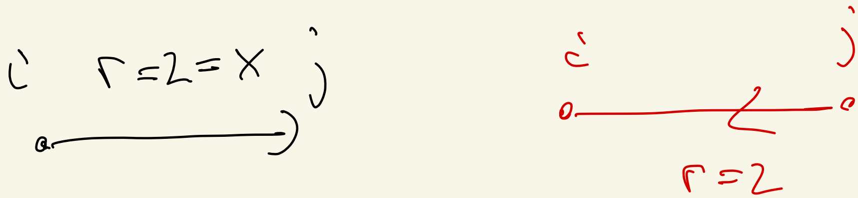


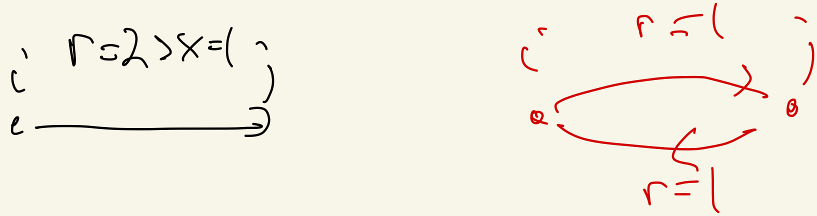
Can when N is just a unit capacity network and $\exists \rightleftarrows$ in N

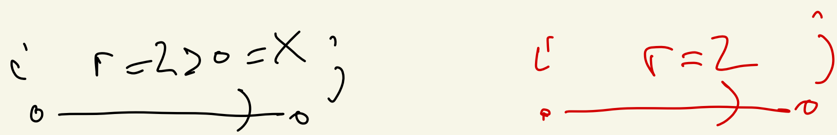


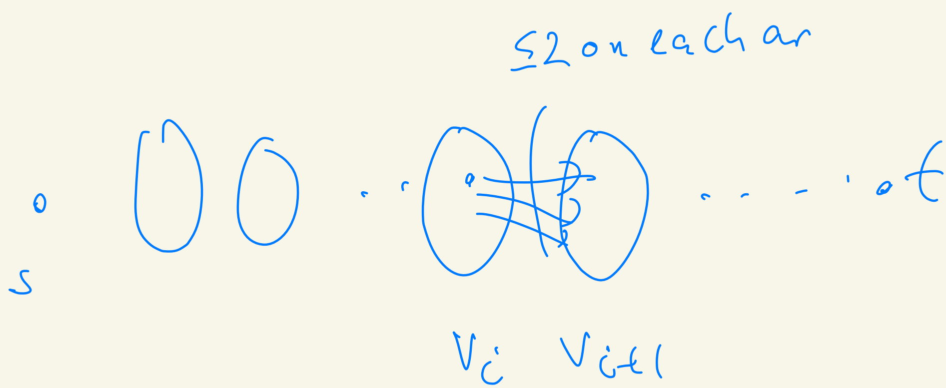
$|x^*| \leq |V_i||V_{i+1}|$


u=1 x=0
i
j
u=1=x
i
r=2
j


i
r=2=x
j
i
j
r=2


i
r=2>x=1
j
r=1
i
j
r=1


i
r=2>0=x
j
i
r=2
j


≤2 on each arc
s
t
V_i
V_{i+1}

$$|X^*| \leq |V_i||V_{i+1}| \cdot 2$$

$$n = |V| \geq \sum |V_i| \geq \sqrt{\frac{|X^*|}{2}} \left(\left\lfloor \frac{\omega+1}{2} \right\rfloor \right)$$

roughly same bound on distance from s to t