Ahuja page 43 on reduced costs

Let $N = (V, A, l, u, b, c)$

a potential on V is a function

$$\pi : V \to \mathbb{R}$$

The reduced cost with respect to π is defined as

$$c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)$$

Suppose x is a feasible flow in N

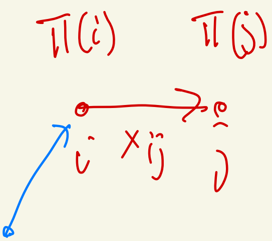
(that is $l_{ij} \le x_{ij} \le u_{ij} \quad \forall ij \in A$

$$b_x(i) = b(i) \quad \forall i \in V)$$

Then we can relate the cost of x wrt c and the cost of x wrt c^{π} as follows:

$$c^{\pi}x = \sum_{ij\in A} c^{\pi}_{ij}\, x_{ij}$$

$$= \sum_{ij\in A} \left(c_{ij} - \pi(i) + \pi(j)\right) x_{ij}$$



$$= \sum_{ij\in A} c_{ij}x_{ij} - \sum_{ij\in A}\left(\pi(i) - \pi(j)\right) x_{ij}$$

$$= cx - \sum_{i\in V} \pi(i)\left(\sum_{ij\in A} x_{ij} - \sum_{ji\in A} x_{ji}\right)$$

$$= cx - \sum_{i\in V} \pi(i)\, b_x(i)$$

$$= cx - \sum_{i\in V} \pi(i) \cdot b(i)$$

$$= cx - \pi b$$

We showed

$$c^{\pi} x = c x - \underbrace{\pi b}_{\text{constant for fixed } \pi}$$

This gives the following very important property

Property 2.4

x is optimal (has minimum cost) w.r.t. c

$\Updownarrow$

x is optimal w.r.t. c^{π}

Property 2.5

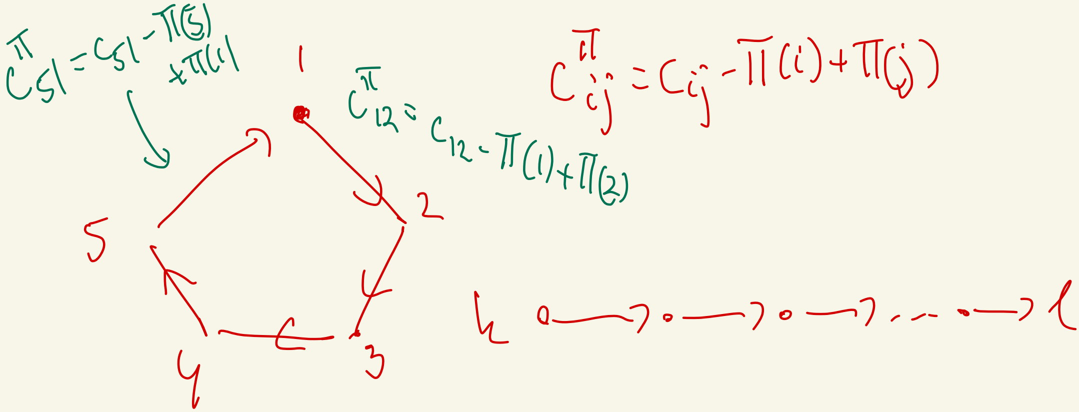
(a) For every cycle W and every potential $\pi : V \to \mathbb{R}$

$$c^{\pi}(W) = c(W)$$

(b) For every (k,ℓ)-path P and every potential $\pi : V \to \mathbb{R}$

$$c^{\pi}(P) = c(P) - \pi(k) + \pi(\ell)$$

$$c^{\pi}_{ij} = c_{ij} - \pi(i) + \pi(j)$$



<u>Corollary</u> If $\pi : V \to \mathbb{R}$ then

(a) $D = (V, A)$ has a negative cycle wrt $c : A \to \mathbb{R}$ if and only if it has a negative cycle wrt the reduced cost function c^{π}

(b) P is a shortest (k, ℓ)-path wrt c if and only if P is a shortest (k, ℓ)-path wrt c^{π}

Example of a potential function

- Suppose $D=(V,A)$ has no negative cycle wrt $c : A \to \mathbb{R}$
- Then we can find shortest path from a vertex s to all other vertices [by the Bellmann-Ford algorithm]
- Let $d(i)$ = length of a shortest (s,i)-path

Then $d(j) \le d(i) + c_{ij}$

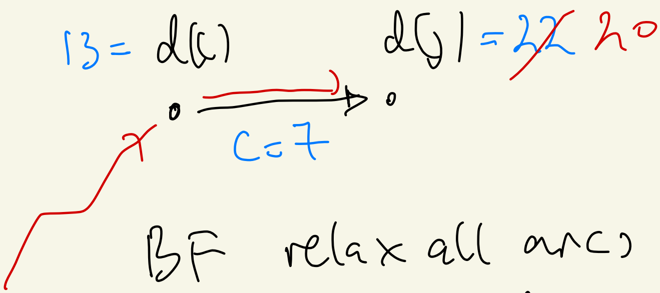
$$i \xrightarrow{c_{ij}} j$$

so $$\underbrace{c_{ij} - (-d(i)) + (-d(j))}_{c^{\pi}} \ge 0$$

c^{π} where $\pi(i) = -d(i)$

Note if D has negative cycle C.

Then we can find one in time $O(nm)$



BF relax all arcs $|V|-1$ times (in any order)

Note that if P is a shortest (s,i)-path wrt c then we have

$$\boxed{d(j) = d(i) + c_{ij} \text{ for every arc } ij \text{ on } P}$$

Hence, if we let $\pi(i) = -d(i)$ then we have $c^{\pi}_{ij} = 0$ for all arcs that are on shortest paths from s

Theorem 9.1 Ahuja

x^* is optimal (has min cost) in $N = (V, A, \ell, u, b, c)$ if and only $N(x^*)$ has no negative cycle

<u>P:</u> If x is feasible and W is a negative cycle in $N(x)$ then $x' = x \oplus \hat{x}$, where $\tilde{x}$ is a cycle flow along W has $cx' < cx$ so x is not optimal

Conversely if x^* is feasible and $N(x^*)$ has no negative cycle then every other feasible flow Z can be obtained as $Z = x^* \oplus \hat{x}$ where $\hat{x}$ is a circulation in $N(x^*)$ and $c\hat{x} \geq 0$ because $\hat{x}$ is the arc sum of (at most m) cycle flows in $N(x^*)$ and each cycle flow has non negative cost

Thus $cZ = cx^* + c\hat{x} \geq cx^*$ so x^* is optimal

Theorem 9.3 in Ahuja Reduced cost optimality conditions

A feasible flow x^* is optimal for $N=(V,A,l,u,b,c)$ if and only if $\exists$ a potential $\pi : V \to \mathbb{R}$ such that $c^{\pi}_{ij} \geq 0 \;\; \forall ij \in N(x^*)$ $\quad (\Delta)$

P: suppose (Δ) holds and let W be any cycle in $N(x^*)$. Then by Property 2.5 $c(W) = c^{\pi}(W)$ and $c^{\pi}(W) \geq 0$ by (Δ) so $N(x^*)$ has no negative cycle $\Rightarrow x^*$ optimal by Thm 9.1

Conversely assume that x^* is optimal and hence $N(x^*)$ has no negative cycle by Thm 9.1

Fix a vertex $s \in V$ and let $d(v)$ denote the length of a shortest (s,v)-path $N(x^*)$ wrt cost fct c

Take $\pi(v) = -d(v)$. Then, as we saw earlier

$$c^{\pi}_{ij} = c_{ij} + d(i) - d(j) \geq 0 \quad \forall ij \in N(x^*)$$

implying that (Δ) holds for this π.

Complementary slackness

Theorem 9.4

A feasible flow x^* in $N=(V,A,\ell,u,b,c)$ is optimal if and only if there exists a potential π such that the following holds for every arc $ij \in A$: (in N)

(a) $c_{ij}^{\pi} > 0 \Rightarrow x_{ij}^* = 0$

(b) $0 < x_{ij}^* < u_{ij} \Rightarrow c_{ij}^{\pi} = 0$

(c) $c_{ij}^{\pi} < 0 \Rightarrow x_{ij}^* = u_{ij}$

$\Downarrow$ suppose x^*, π satisfy $(\triangle)$

Then (a), (b), (c) must hold

$\Uparrow$ suppose (a)–(c) hold then

x^*, π must satisfy $(\triangle)$ so x^* is optimal