Algorithms for finding min cost flows

Thm 3.10 BJG = 9.1 in Ahujs

A feasible flow x in $N = (V, A, l, u, b, c)$ is optimal (min cost) if and only if $N(x)$ has no negative cycle.

This leads to

Cycle cancelling algorithm

1. Find a feasible flow x in N (stop if no such)
2. while $N(x)$ has a negative cycle W

   let δ be capacity of W

   $x \leftarrow x \oplus \delta(W)$

3. Return x

Theorem 3.10.2

If ℓ, u, c and b are all integer valued

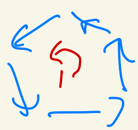
Then the cycle cancelling algorithm finds an optimal flow x in time

$O(nm^2 CU)$ when $C = \max\limits_{ij \in A} |c_{ij}|$

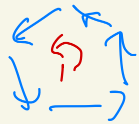
$U = \max\limits_{ij \in A} u_{ij}$

- max possible cost of a flow in N is mCU
- min possible cost of a flow in N is $-mCU$

$$mCU \quad \text{decrease by at least } 1$$
$$\downarrow$$
$$-mCU$$



mCU decrease by at least 1



$-mCU$

$O(mCU)$ iterations (finding a cycle in $N(x)$)

time to check for negative cycle and find out if it exists $O(nm)$ by Bellmann-Ford.

Total complexity

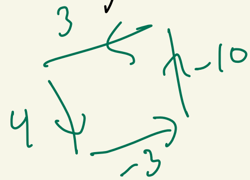
$$O(nm) \cdot O(mCU)$$

$$= O(nm^2CU)$$

Theorem 3.10.3 Integrality theorem for min cost flow

If l, u, b, c are all integer valued then $\exists$ an integer valued optimal flow x

The **mean cost** of a cycle W is $\dfrac{c(W)}{|A(W)|}$



$\dfrac{-6}{4} = \dfrac{-3}{2}$

Theorem If the cycle cancelling algorithm always augments maximally along a cycle of minimum mean cost then it runs in time $O(n^2 m^3 \log n)$ even if some arcs have irrational data (l, u or c)

Theorem 3.10.5 (Buildup theorem)

Let x be an optimal flow with respect to $b = b_x$ (and $\ell \le x \le u$).

If P is a shortest (p,q)-path in $N(x)$ and $\alpha \le \delta(P)$

Then the flow $x' = x \oplus \alpha P$ is optimal wrt the balance function b'

with $b'(v) = \begin{cases} b(v) & \text{if } v \ne p,q \\ b(v)+\alpha & \text{if } v = p \\ b(v)-\alpha & \text{if } v = q \end{cases}$

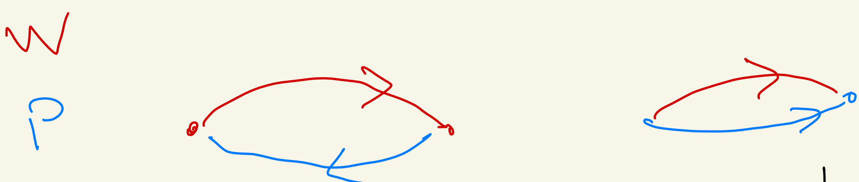
Proof: It suffices to show that $N(x')$ has no negative cycle

Suppose W negative cycle in $N(x')$

Recall that each arc in $N(x')$ is either also in $N(x)$ or it is opposite to an arc $i \to j$ in $N(x)$ and then P contains the arc $i \to j$

Consider the multidigraph H defined by $H = (V, \tilde{A})$ where

$\tilde{A}$:

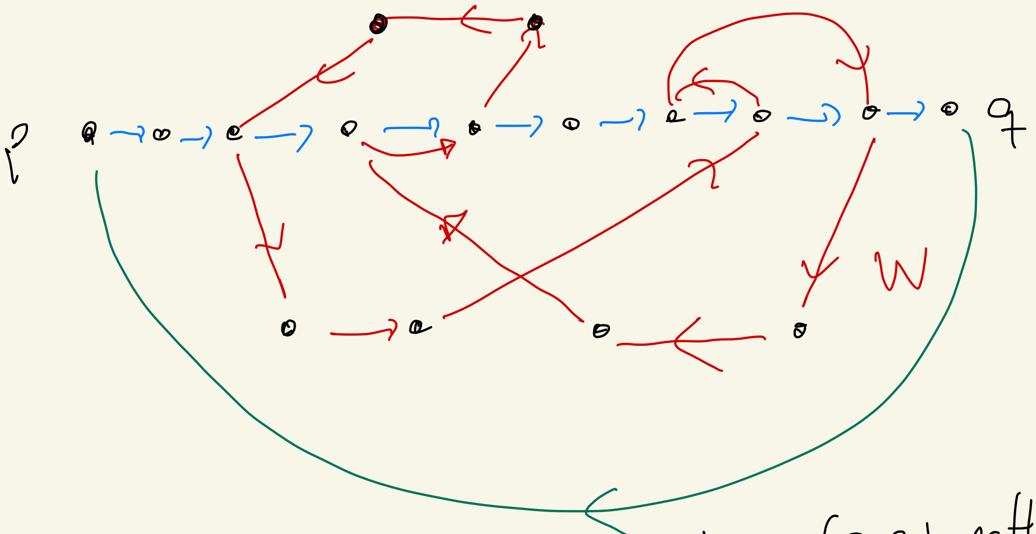


cancel out

keep both

$H + \{qp\}$ is eulerian

$H + \{qp\}$ is eulerian



Hence H decomposes into a (p,q)-path P' and some arc-disjoint cycles $W_1, W_2, \ldots W_k$ (by BJG ex 3.8)

- Each W_i is a cycle in $N(x)$

cost of opposite arcs cancel out ($c_{ij} = -c_{ji}$) so as $c(W) < 0$

$$c(P) > c(P) + c(W)$$

$$= c(P') + \sum_{i=1}^{k} c(W_i)$$

$$\geq c(P')$$

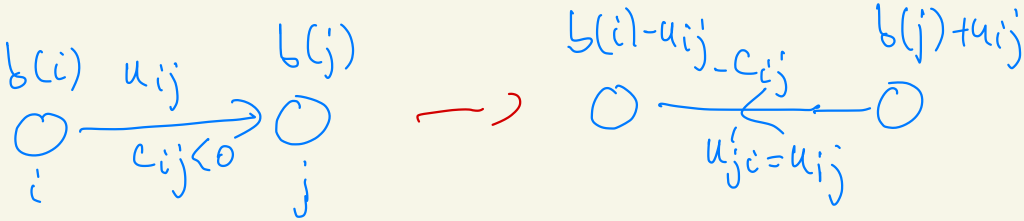
as each $c(W_i) \geq 0$ by optimality of x

Thus $c(P) > c(P')$ $\Big\}$ as P was a shortest (p,q)-path

BJG exercise 3.49 and Ahuja page 40:

We can always modify a network $N=(V,A,\ell\equiv 0,u,b,c)$ with $c_{ij}<0$ for some arc $ij\in A$ to a network $N'=(V,A',\ell\equiv 0,u',b',c)$ so that given an optimal flow x' in N' we can obtain an optimal flow x in N

Reverse arc with $c_{ij}<0$:



$$x_{ij}=u_{ij}-x'_{ij} \longleftarrow x'_{ij}$$

Consequence: we may assume that $c_{ij} \geq 0 \ \forall ij \in A$

Then $x \equiv 0$ is optimal (min cost among all flows with $b_x(v) = 0 \ \forall v$) since $N(x) = N$ has no negative cycle

For a given optimal flow x and $N = (V, A, l \equiv 0, u, b, c)$ set

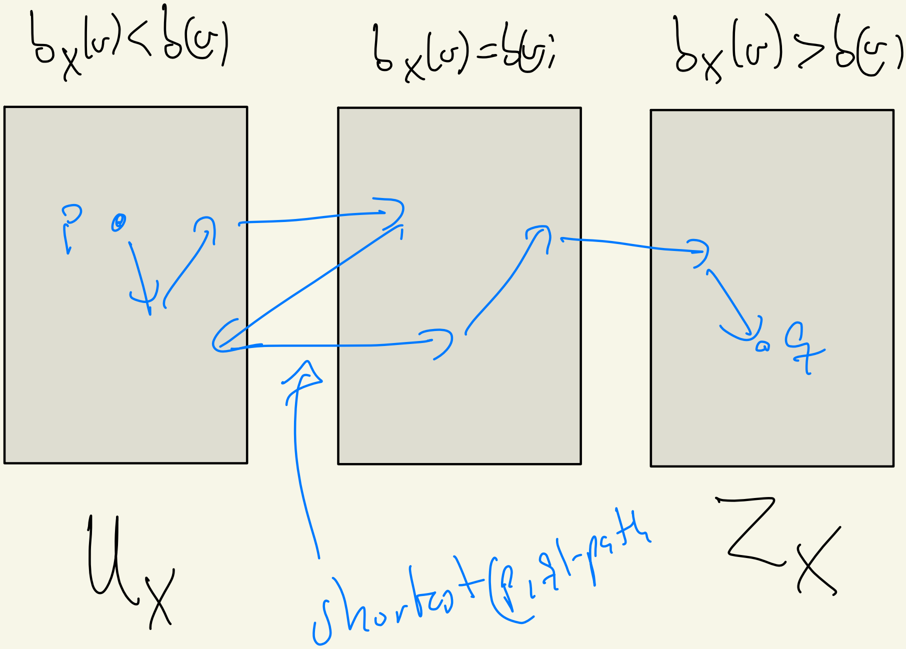
$$U_x = \{ v \mid b_x(v) < b(v) \}$$

$$Z_x = \{ v \mid b_x(v) > b(v) \}$$

note $U_x = \emptyset \Leftrightarrow Z_x = \emptyset$

and then x is feasible and optimal flow in N

For a given optimal x:



Buildup algorithm:

1. $x_{ij} \leftarrow 0 \quad \forall ij \in A$; Find U_x, Z_x
2. If $U_x = \emptyset$ goto 8.
3. If $\nexists\, (U_x, Z_x)$-path in $N(x)$ goto 9.
4. choose $p \in U_x$, $q \in Z_x$ s.t $N(x)$ has a (p,q)-path
5. Find a shortest (p,q)-path P in $N(x)$
6. $\varepsilon \leftarrow \min\{\delta(P), b(p)-b_x(p), b_x(q)-b(q)\}$
7. $x \leftarrow x \oplus \varepsilon P$; Find new U_x, Z_x and go to 2
8. Return x
9. Return 'no feasible solution'

Theorem 3.10.6

Let $N = (V, A, l \equiv 0, u, b, c)$ have integer data and $c_{ij} \geq 0 \ \forall ij \in A$

Then the buildup algorithm constructs an minimum cost flow in N or determines that no feasible flow exists in N.

The running time is $O(n^2 m M)$

when $M = \max_{v \in V} |b(v)|$

- No vertex has its balance increased more than M times.
- To find new shortest path takes $O(nm)$
- At most $n \cdot M$ augmentation steps

$\Rightarrow O(n^2 m M)$

Why does there exist a (U_x, Z_x)-path in $N(x)$ if N has a feasible flow?

suppose y is feasible in $N = (V, A, \ell \equiv 0, u, b, c)$

$b_y(v) = b(v) \quad \forall v$

$y = x \oplus x''$ $\quad x''$ flow in $N(x)$

$b(v) = b_y(v) = b_x(v) + b_{x''}(v)$

For $p \in U_x$: $b_{x''}(p) > 0$

$q \in Z_x$: $b_{x''}(q) < 0$

$b(v) = b_y(v) = b_x(v) + b_{x''}(v)$

For $p \in U_x$: $b_{x''}(p) > 0$

$q \in Z_x$: $b_{x''}(q) < 0$

$v \notin U_x \cup Z_x$: $b_{x''}(v) = 0$

Decompose x'' into path + cycle flows

all paths start in U_x and ends in Z_x

In particular $\exists$ a (U_x, Z_x)-path !! in $N(x)$