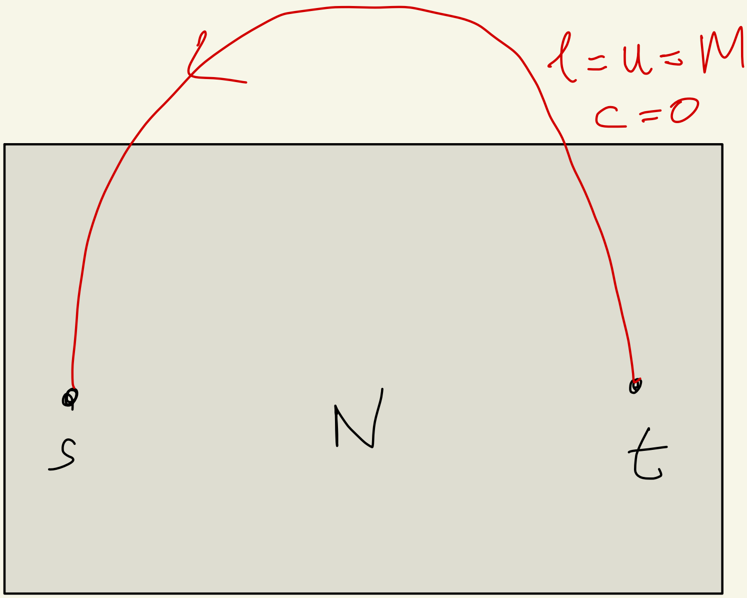
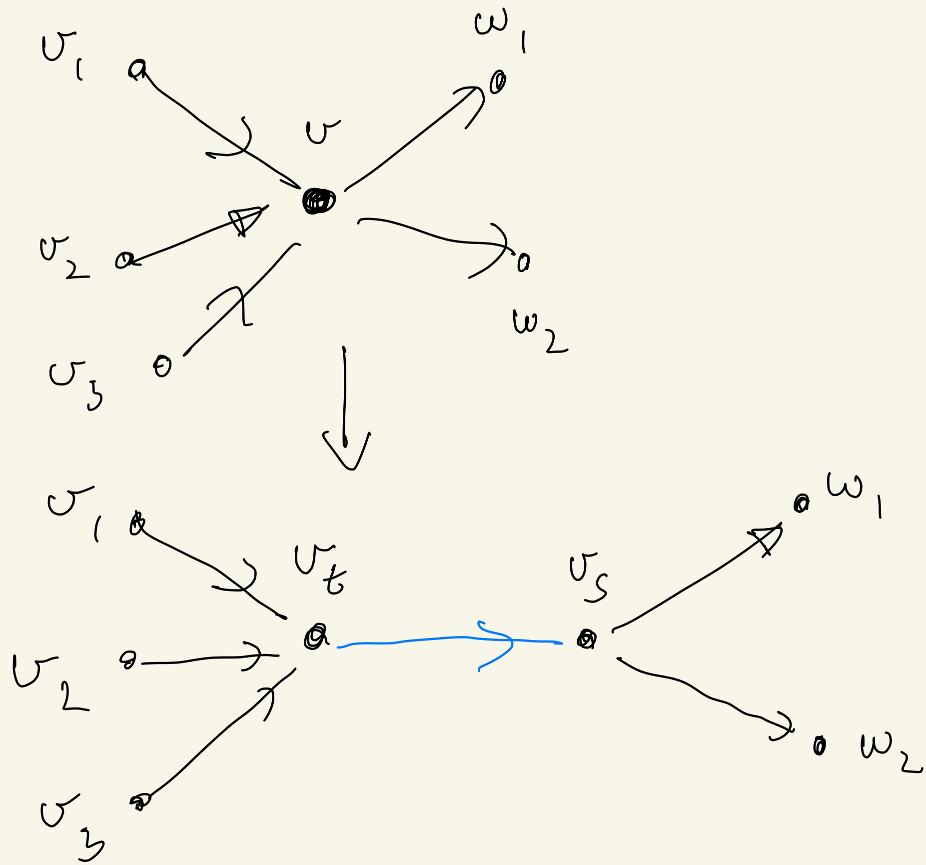
(s,t)-flows $\longrightarrow$ circulations

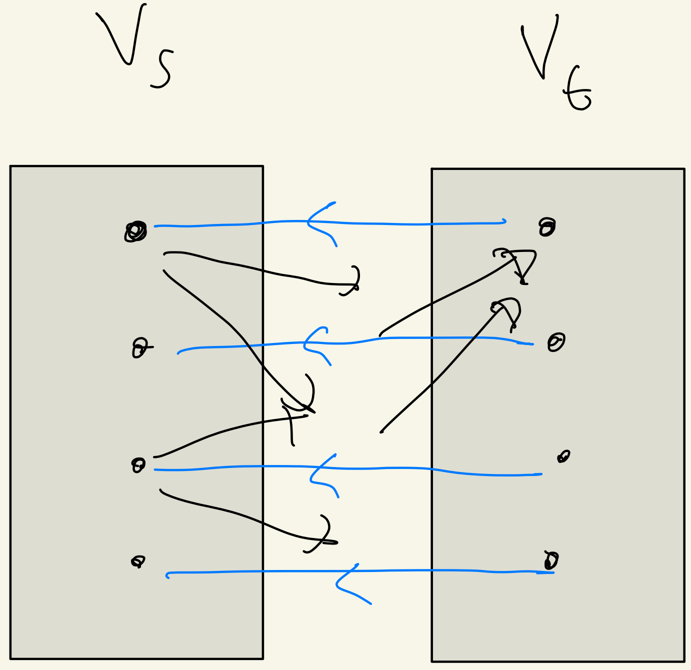
$b_X(v) = 0$



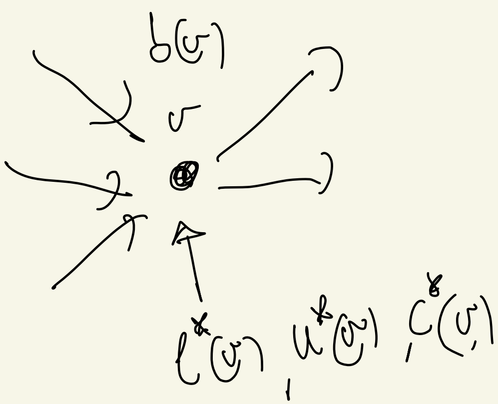
in N $b(s) = M = -b(t)$

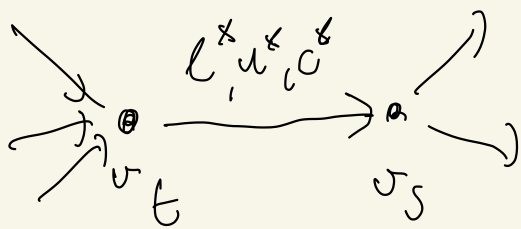
vertex splitting:





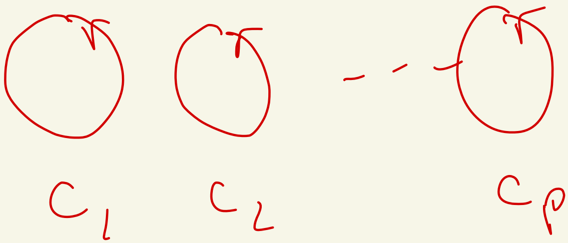
Handling bounds and cost on vertices





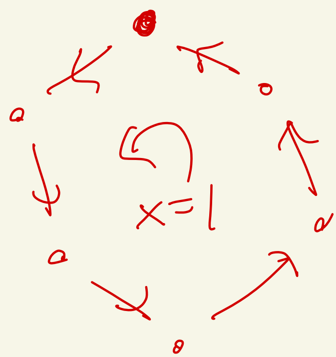
Example:

Cycle factor in a digraph $D=(V,A)$



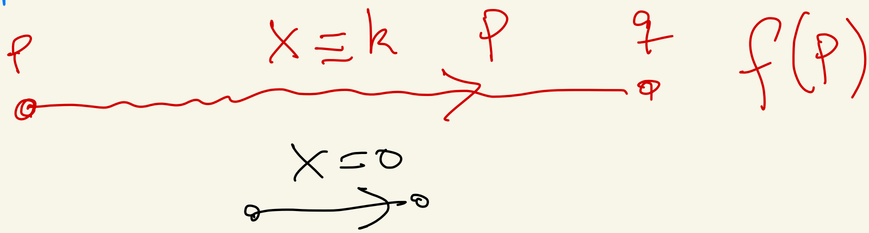
$$\bigcup_{i\in[p]} V(C_i) = V$$

$$\ell^{\alpha} = u^{\alpha} = 1$$

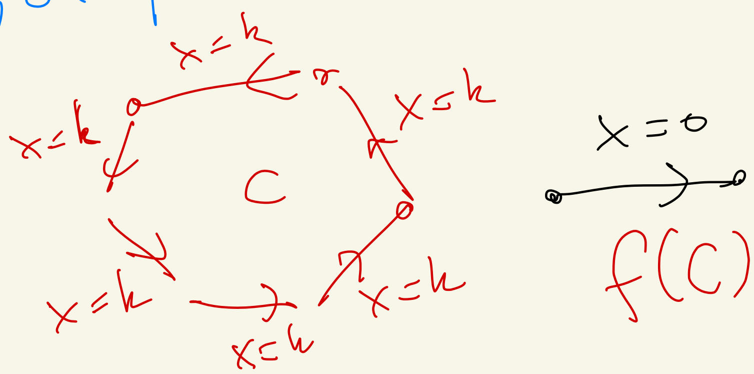


Flow decomposition

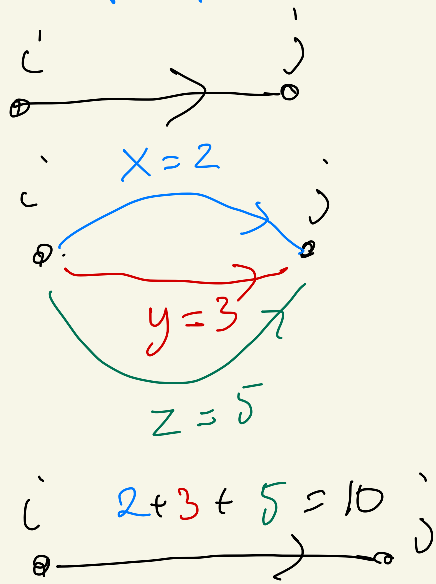
- path flow



- cycle flow



arc sum of flows



<u>Thm 3.3.1</u> Let x be a flow in N

Then x is the arc sum of some path flows

$f(P_1), \ldots f(P_\alpha)$ and

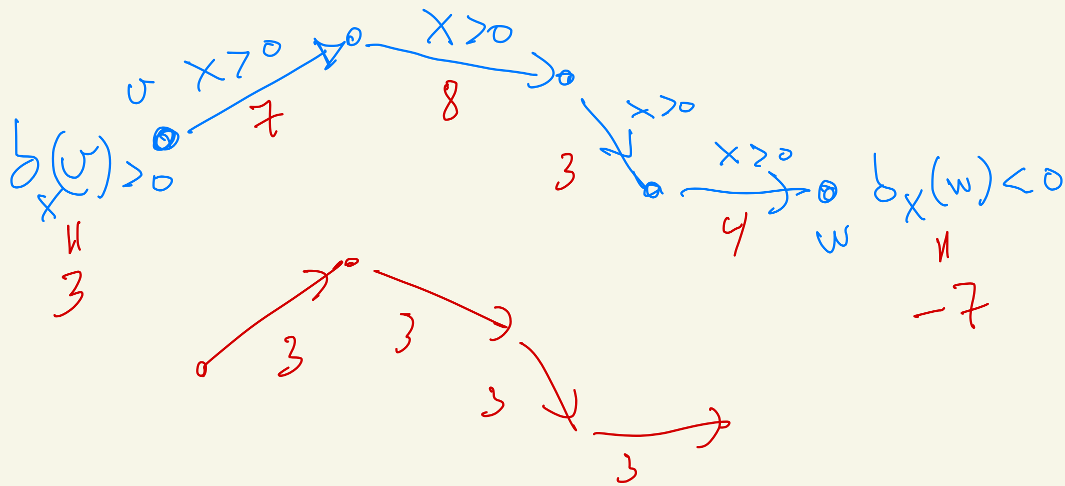
cycle flows $f(C_1), \ldots, f(C_\beta)$

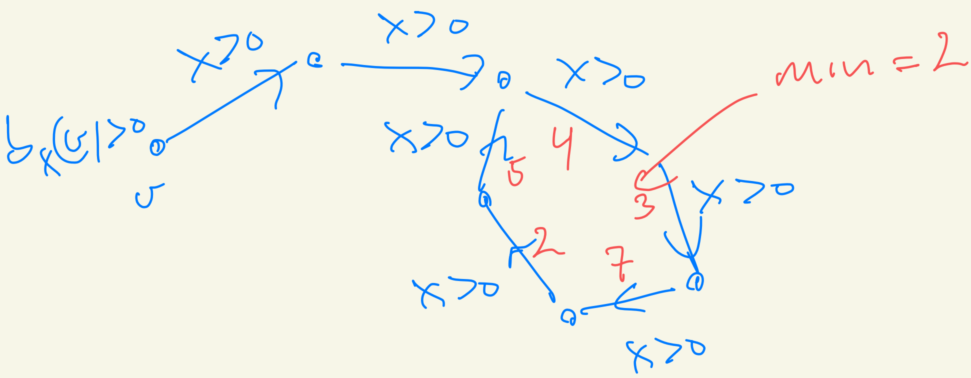
s.t. (a) Each P_i joins a source to a sink

(b) $\alpha + \beta \leq n + m$ and $\beta \leq m$

P: let x be a flow and $x \not\equiv 0$

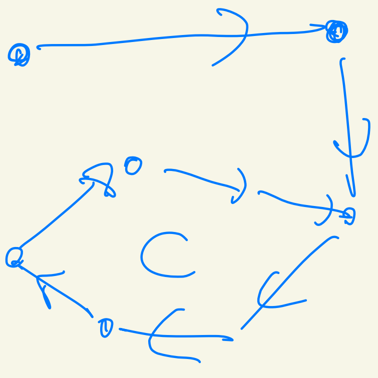
Can I $\exists\, v$ s.t $b_x(v) > 0$





Case 2 $\quad \delta_x(v) = 0 \;\; \forall v \in V$

$x > 0$



$\square$

Corollary

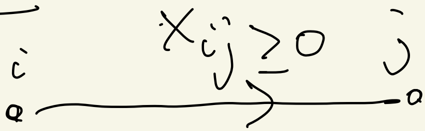
Every circulation decomposes into at most m cycle flows

Working with the residual network $N(x)$

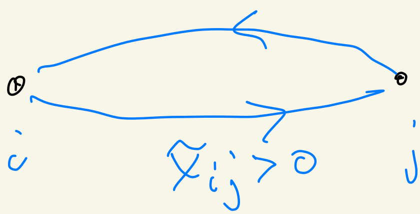
'adding' a residual flow (flow in $N(x)$) to x

suppose $\tilde{x}$ is feasible netto flow in $N(x)$

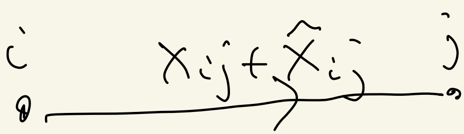
$\underline{x \oplus \tilde{x}}$:



in N



in $N(x)$



$$i \xrightarrow{x_{ij} \geq 0} j$$

$r_{ij} = (u_{ij} - x_{ij}) +$

$(x_{ji} - l_{ji})$

$$j \xleftarrow[\tilde{x}_{ji} > 0]{} i$$

if $\quad x_{ij} \geq \tilde{x}_{ji} \qquad i \xrightarrow{x_{ij} - \tilde{x}_{ji}} j$

if $\quad x_{ij} < \tilde{x}_{ji} \qquad i \xleftarrow{\tilde{x}_{ji} - x_{ij}} j$

<u>Thm 3.4.2</u> (check yourselves)

$\forall \tilde{x} \in N(x) \quad \bar{x} = x \oplus \tilde{x}$ is feasible in N

and $\quad b_{\bar{x}}(v) = b_x(v) + b_{\tilde{x}}(v)$

$$c^T \bar{x} = c^T x + c^T \tilde{x}$$

Thm 3.4.3

Let x and x' be feasible in

$N = (V, A, \ell \equiv 0, u, c)$

Then there exists a feasible flow

$\bar{x}$ in $N(x)$ such that

$$x' = x \oplus \bar{x}$$

$\left(\text{and } b_{\bar{x}}(v) = b_x(v) - b_{x'}(v)\right)$

In particular if $b_x \equiv b_{x'}$

then $b_{\bar{x}} \equiv 0$ so $\bar{x}$ is a circulation

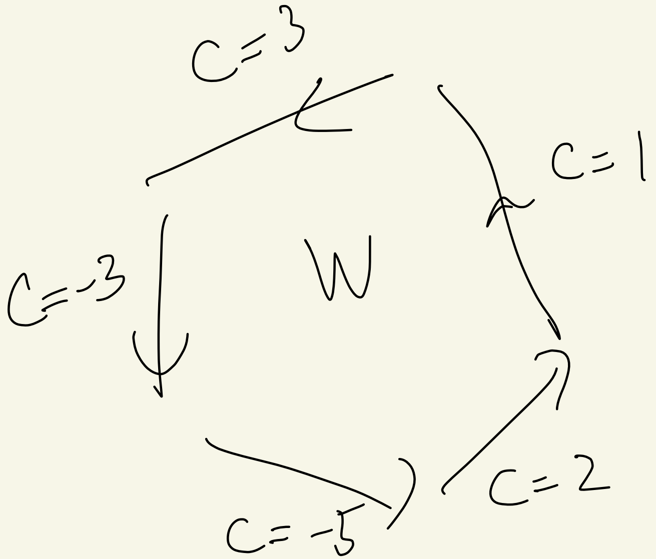
Cor. 3.4.4 if x, x' feasible in $N = (V, A, \ell \equiv 0, u, b, c)$

then there exist cycles $W_1, W_2, \ldots, W_k$ $\boxed{k \leq m}$

such that

(a) $x' = x \oplus (\underbrace{f(W_1) + \ldots + f(W_k)}_{\bar{x}})$

(b) $c^T x' = c^T x + \sum_{i=1}^{k} \text{cost}(f(W_i))$



$$c(W) = -2$$

So <u>if</u> x' has minimum cost among all feasible flows in N then either x also has minimum cost or $\exists$ cycle W in $N(x)$ with $c(W) < 0$

Thm

a flow x is of minimum cost in $N=(V,A,l\equiv 0,u,b,c)$ if and only if there is no negative cycle in $N(x)$

$W \in N(x)$

$c(W)<0$ $\Rightarrow$ x not minimum cost

Cycle cancelling 'algorithm'

1. Find a feasible flow x in

$$N = (V, A, \ell \equiv 0, u, b, c)$$

2. while $N(x)$ has a negative cycle

send flow around such a cycle W